Eidgenössische Technische Hochschule Zürich

Analysis II: Multiple Variables

Spring Semester 2026

Lecture Notes

Contents

References

[1] Serra, Joaquim: Skript zur Vorlesung Analysis II. ETH Zürich. 2026

Analysis II: Multiple Variables

Fynn Krebser–fkrebser@student.ethz.ch

Spring Semester 2026

Lec 1

1 Metric Spaces

In analysis II, will be working mostly in $\mathbb{R}^n$, which is a vector space defined as

$$\mathbb{R}^n := \{x = (x_1, \dots, x_n), x_i \in \mathbb{R}\}.$$

Here, we call $n \in \mathbb{N}$ the DIMENSION. We want to add some extra structure in particular the EUCLIDEAN STRUCTURE

Definition 1.1: Euclidean structure

On $\mathbb{R}^n$ we define the EUCLIDEAN NORM as

$$\|x\| := \sqrt{x_1^2 + \cdots + x_n^2},$$

which describes the length of the vector x. We also define the SCALAR PRODUCT as

$$x \cdot y = \langle x, y \rangle := \sum_{i=1}^{n} x_i y_i,$$

furthermore the EUCLIDEAN DISTANCE is defined as

$$d(x, y) := \|x - y\|.$$

Lemma 1.2: Triangle inequality

For all $x, y, z \in \mathbb{R}^n$ we have

$$\|z - x\| \leq \|y - x\| + \|z - y\|.$$

The lemma states that the shortest path between two points is a straight line.

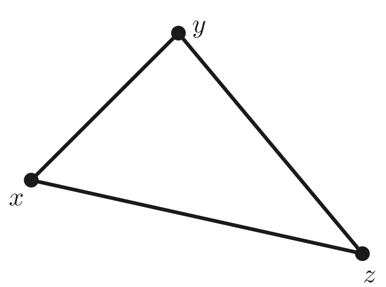

Figure 1: Triangle Inequality

Proof. Equivalently, $\|a + b\| \leq \|a\| + \|b\|$ for all $a, b \in \mathbb{R}^n$. This is because if we let $a = y - x$ and $b = z - y$, then $a + b = z - x$. Equivalently, squaring both sides, we have

$$\|a + b\|^2 \leq \|a\|^2 + \|b\|^2 + 2\|a\|\|b\|. \tag{1.1}$$

By definition of the norm, we have

$$\sum_{i=1}^{n} (a_i + b_i)^2 = \sum_{i=1}^{n} a_i^2 + b_i^2 + 2a_i b_i = \|a\|^2 + \|b\|^2 + 2a \cdot b.$$

Together with (1.1), we have that our statement is equivalent to $a \cdot b \leq \|a\|\|b\|$, which is the Cauchy-Schwarz inequality. □

Lemma 1.3: Cauchy-Schwarz inequality

For all $x, y \in \mathbb{R}^n$ we have

$$x \cdot y \leq \|x\|\|y\|.$$

Proof. If either $x = 0$ or $y = 0$, then the statement becomes $0 \leq 0$, which is true.

Let now $x, y \in \mathbb{R}^n$ be nonzero. Now for every $\lambda > 0$, we have

$$2x \cdot y = 2\sum_{i=1}^{n} \lambda x_i \frac{y_i}{\lambda} \leq \sum_{i=1}^{n} \lambda^2 x_i^2 + \frac{y_i^2}{\lambda^2} = \lambda^2\|x\|^2 + \frac{1}{\lambda^2}\|y\|^2,$$

Since $2ab \leq a^2 + b^2$. Since x, y are nonzero we can take $\lambda^2 = \frac{\|y\|}{\|x\|}$, which gives us

$$2x \cdot y \leq 2\|x\|\|y\|.$$

□

In order to not always define convergence and continuity in other subjects, we will now introduce the notion of a METRIC SPACE, which is a set with a distance function defined on it.

Definition 1.4: Metric space

A metric space is a pair (X, d), where X is a set and $d : X \times X \to [0, \infty)$ is a function such that for all $x, y, z \in X$ we have

- $d(x, y) = 0$ if and only if $x = y$,
- $d(x, y) = d(y, x)$, and
- $d(x, z) \leq d(x, y) + d(y, z)$.

Example 1.5:

1. $(\mathbb{R}^n, d_{\text{Euclidean}})$ is a metric space.
2. $(\mathbb{R}^2, d_{\text{NY}}(x, y))$ is a metric space, where the MANHATTAN METRIC is defined as

$$d_{\text{NY}}(x, y) = |x_1 - y_1| + |x_2 - y_2|.$$

3. Given (X, d) a metric space and $Y \subseteq X$, then $(Y, d|_{Y \times Y})$ is a metric space.

For example if we take $X = \mathbb{R}^2$ and Y to be the unit circle, then we also have the metric space $(Y, d_{\text{Euclidean}}|_{Y \times Y})$.

In general it is useful to first think about the proof in $\mathbb{R}^n$, and then to see if it can be adapted to a more general metric space. This is often easier as we tend to have a better intuition for $\mathbb{R}^n$.

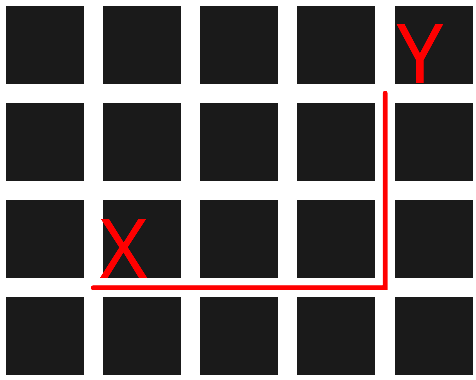

Figure 2: Manhattan (Glarus) metric visualized

Given $a < b \in \mathbb{R}$, and

$$X = \{f : [a, b] \to \mathbb{R}, f \text{ continuous }\},$$

then how can we find the distance between two functions $f, g \in X$? It can for example be defined as

$$d(f, g) = \max_{[a,b]} |f(x) - g(x)|.$$

Another distance would be

$$d(f, g) = \sqrt{\int_a^b |f(x) - g(x)|^2 dx}.$$

Lec 2 As an exercise, proof that both of these are metrics on X.

In the following, let (X, d) be a metric space. We want to define the notion of convergence and continuity in X.

Definition 1.6: Sequence

Given a set X, we call a map $x : \mathbb{N} \to X$ a SEQUENCE in X. Instead of writing $x(n)$, we write x_n as the n-th term of the sequence $\in X$. To denote the full sequence we write $(x_n)_{n\geq 0}$.

Definition 1.7: Convergent Sequence

We say $(x_n)_{n\geq 0}$ has limit $x \in X$ if $d(x_n, x) \to 0$ as real numbers. x is called the LIMIT of the sequence, and we write $\lim\limits_{n\to\infty} x_n = x$.

Another way to phrase this is that for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$ we have $d(x_n, x) < \varepsilon$.

Notice that the limit has to be in the set unlike in Analysis I, as otherwise the metric is not defined.

Lemma 1.8: Uniqueness of the limit

Given a sequence $(x_n)_{n\geq 0}$ such that $x_n \to x$ and $x_n \to y$, then $x = y$.

Proof. Assume by contradiction that $x \neq y$ and let $3\varepsilon = d(x, y) > 0$. By the definition of the limit, $\exists N_x$ such that $d(x_n, x) < \varepsilon \forall n \geq N_x$, and $\exists N_y$ such that $d(x_n, y) < \varepsilon \forall n \geq N_y$.

Let $N = \max(N_x, N_y)$, then for all $n \geq N$ we have

$$3\varepsilon = d(x, y) \leq d(x, x_n) + d(x_n, y) < 2\varepsilon,$$

by triangle inequality, which is a contradiction. □

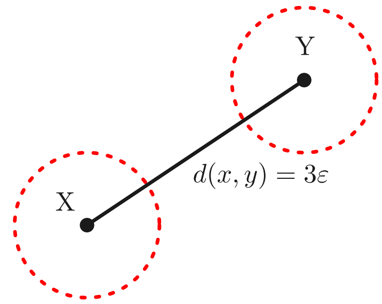

Figure 3: Uniqueness of the limit

Definition 1.9: Subsequence

Given a sequence $(x_n)_{n\geq 0}$, we define a SUBSEQUENCE as a sequence of the form

$$(x_{f(k)})_{k\geq 0},$$

where $f : \mathbb{N} \to \mathbb{N}$ is a strictly increasing function.

Usually, we will write x_{n_k} instead of $x_{f(k)}$ for the subsequence.

Definition 1.10: Accumulation Point

Given $Y \subset X$, we say that $y \in X$ is an ACCUMULATION POINT of Y if there exists a sequence $(y_n)_{n\geq 0}$ in Y such that $y_n \to y$.

Given $(x_n)_{n\geq 0}$ as a sequence of X we say that x is an ACCUMULATION POINT of $(x_n)_{n\geq 0}$ if there exists a subsequence $(x_{n_k})_{k\geq 0}$ such that $x_{n_k} \to x$.

Lemma 1.11:

Given a sequence $(x_n)_{n\geq 0}$. The sequence converges to x if and only if $\forall (x_{n_k})_{k\geq 0}$ we have $x_{n_k} \to x$.

Proof. We rewrite the statement to $x_n \to x \Leftrightarrow (x_{n_k}) \to y \Rightarrow y = x$.
$\Leftarrow$: Taking the particular subsequence $x_{n_k} = x_n$, we have $x_n \to y$, and thus $y = x$.

$\Rightarrow$: Let $f : \mathbb{N} \to \mathbb{N}$ be a strictly increasing function. Since $x_n \to x$, for every $\varepsilon > 0$ there exists N such that for all $n \geq N$ we have $d(x_n, x) < \varepsilon$. Since $f(n)$ is increasing, $f(n) \geq n$, thus for all $n \geq N$ we have $d(x_{f(n)}, x) < \varepsilon$, which means that $x_{f(n)} \to x$. □

Lemma 1.12:

Given a sequence $(x_n)_{n\geq 0}$. Then $x_n \to x$ if and only if every subsequence $(x_{n_k})_{k\geq 0}$ has a subsequence $(x_{n_{k_l}})_{l\geq 0}$ such that $x_{n_{k_l}} \to x$.

Proof. See later... We will do it later try first at home. □

Definition 1.13: Cauchy Sequence

We say that a sequence $(x_n)_{n\geq 0}$ is a CAUCHY SEQUENCE if $\forall \varepsilon > 0$, $\exists N$ such that $d(x_n, x_m) < \varepsilon$ whenever $n, m \geq N$.

Definition 1.14: Complete Metric Space

We say that a metric space (X, d) is COMPLETE if every Cauchy sequence in X converges (to a limit in X).

Example 1.15:

1. $(\mathbb{R}, d_{\text{Euclidean}})$ is a complete metric space.

Theorem 1.16:

$(\mathbb{R}^n, d_{\text{Euclidean}})$ is complete.

Lemma 1.17:

Given a sequence $(x_m)_{m\geq 0} \subset \mathbb{R}^n$, then $x_m \to x \in \mathbb{R}^n$ if and only if $x_{m,i} \to x_i$ for all $i = 1, \dots, n$, where $x_m = (x_{m,1}, \dots, x_{m,n})$ and $x = (x_1, \dots, x_n)$.

Proof. $\Rightarrow$: Since $x_m \to x$, $\forall \varepsilon > 0 \exists N$ such that $\|x_n - x\| < \varepsilon$ for all $n \geq N$. In particular,

$$|x_{m,i} - x_i| \leq \sqrt{\sum_{j=1}^{n} |x_{m,j} - x_j|^2} = \|x_m - x\| < \varepsilon.$$

$\Leftarrow$: Since $x_{m,i} \to x_i$ for all $i = 1, \dots, n$, $\forall \varepsilon > 0 \exists N_i$ such that $|x_{m,i} - x_i| < \frac{\varepsilon}{\sqrt{n}}$ for all $m \geq N_i$. Let $N = \max(N_1, \dots, N_n)$, then for all $m \geq N$ we have

$$\|x_m - x\| = \sqrt{\sum_{i=1}^{n} |x_{m,i} - x_i|^2} < \sqrt{\sum_{i=1}^{n} \frac{\varepsilon^2}{n}} = \varepsilon.$$

□

Proof. [Proof of Theorem 1.16] Given $(x_m)_{m\geq 0}$ Cauchy, let us show $\exists x$ such that $x_m \to x$.

By Lemma 1.17, it suffices to show that $\forall i = 1, \dots, n$ we have that $(x_{m,i})_{m\geq 0}$ is Cauchy in $\mathbb{R}$.

But since Cauchy sequences in $\mathbb{R}$ converge, $\exists x_i$ such that $x_{m,i} \to x_i$ for all $i = 1, \dots, n$. Thus by Lemma 1.17, we have $x_m \to x$, where $x = (x_1, \dots, x_n)$.

□

Tip 1.18:

An often seen counter example is the discrete metric space, where $d(x, y) = 1$ if $x \neq y$ and 0 otherwise.

Lec 3

1.1 Topology of metric spaces

We would like to define the notion of open sets in a metric space, which will allow us to define continuity and other topological properties of metric spaces.

Definition 1.19: **Open Ball**

Define the OPEN BALL of radius $r > 0$ centered at $x \in X$ as

$$B_r(x) = B(x, r) := \{y \in X : d(x, y) < r\}.$$

Definition 1.20: **Open and Closed Sets**

We say that $U \subset X$ is OPEN if for every $x \in U, \exists r > 0$ such that $B(x, r) \subset U$.

$A \subset X$ is CLOSED, if $X \setminus A$ is open.

Exercise 1.21:

Show that $(0, 1) \times (0, 1) \subset \mathbb{R}^2$ is open.

Show that $[0, 1] \times [0, 1] \subset \mathbb{R}^2$ is closed.

The topology associated to d is

$$\mathcal{T} = \{U \subset X : U \text{ is open}\}.$$

Lemma 1.22:

Arbitrary unions of open sets are open.

$$\{U_i\}_{i\in\mathcal{I}}, U_i \subset X \text{ open} \Rightarrow \bigcup_{i\in\mathcal{I}} U_i \text{ is open.}$$

Finite intersections of open sets are open.

$$U_1, \dots, U_k \subset X \text{ open} \Rightarrow \bigcap_{i=1}^{k} U_i \text{ is open.}$$

Proof. Take $x \in \bigcup_{i\in I} U_i$, then $x \in U_j$ for some $j \in I$. Since U_j is open, $\exists r > 0$ such that $B(x, r) \subset U_j \subset \bigcup_{i\in I} U_i$, thus $\bigcup_{i\in I} U_i$ is open.

Take $x \in \bigcap_{i=1}^{k} U_i$, then $x \in U_i$ for all $i = 1, \dots, k$. Since U_i is open, $\exists r_i > 0$ such that $B(x, r_i) \subset U_i$ for all $i = 1, \dots, k$. Let $r = \min(r_1, \dots, r_k)$, then $B(x, r) \subset B(x, r_i) \subset U_i$ for all $i = 1, \dots, k$, thus $\bigcap_{i=1}^{k} U_i$ is open. □

Lemma 1.23:

Finite unions of closed sets are closed.

Arbitrary intersections of closed sets are closed.

Proof. Notice that $X \setminus \bigcup_{i=1}^{k} A_i = \bigcap_{i=1}^{k} (X \setminus A_i)$, and $X \setminus A_i$ is open for all $i = 1, \dots, k$, thus $\bigcup_{i=1}^{k} A_i$ is closed. □

Example 1.24: **Finite is Important**

Take $(\mathbb{R}, d_{\text{Euclidean}})$ and $U_k = (-\frac{1}{k}, \frac{1}{k})$. Then

$$\bigcap_{k=1}^{\infty} U_k = \{0\}.$$

Which is not open.

Definition 1.25: **Interior, Closure and Boundary**

Given $\Omega \subset X$, we define

- $\text{int}\Omega = \Omega^\circ = \{U \subset \Omega \mid U \text{ is open }\}$ as the INTERIOR of Ω,
- $\overline{\Omega} = \{x \in X \mid \exists (x_n)_{n\geq 0} \subset \Omega, x_n \to x\}$ as the CLOSURE of Ω, and
- $\partial\Omega = \overline{\Omega} \setminus \Omega^\circ$ as the BOUNDARY of Ω.

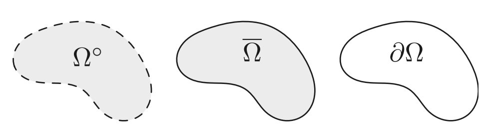

Figure 4: Interior, closure and boundary of a set Ω

Lemma 1.26:

$U \subset X$ is open if and only if whenever $(x_n)_{n\geq 0} \subset X$ such that $x_n \to x \in U$, then $x_n \in U$ eventually.

$A \subset X$ is closed if and only if whenever $(x_n)_{n\geq 0} \subset A$ such that $x_n \to x$, then $x \in A$.

Proof. 1, $\Rightarrow$: Take $x \in U$. By definition of open set, $\exists r > 0$ such that $B(x, r) \subset U$. Since $x_n \to x$, $\exists N$ such that $x_n \in B(x, r) \forall n \geq N$. (Since this is equivalent to $d(x_n, x) < r$ for all $n \geq N$). Thus $x_n \in U$ eventually.

1, $\Leftarrow$: We argue by contraposition. Since U is not open, $\exists x \in U$ such that $\forall r > 0, B(x, r) \not\subset U$. This is the same as saying $\exists x_r \in B(x, r) \cap X \setminus U$.

Taking $r = \frac{1}{n}$ and calling $x_n = x_{\frac{1}{n}} \to x$. Since $x_n \in X \setminus U$ for all $n \geq 0$, we have that $x_n \to x \in U$ but $x_n \notin U$ for all $n \geq 0$, contradicting that $x_n \in U$ eventually.

2. Exercise □

Exercise 1.27:

Given (X, d) complete, $A \subset X$ closed, show that (A, d) is complete.

Definition 1.28: Continuity

Given $f : X \to Y$ with (X, d_X) and (Y, d_Y) metric spaces, we say that f is CONTINUOUS if one of the following 3 equivalent properties hold:

- Epsilon-Delta Continuity: $\forall x \in X, \forall \varepsilon > 0, \exists \delta > 0$ such that:

$$\forall x' \in X, d_X(x, x') < \delta \Rightarrow d_Y(f(x), f(x')) < \varepsilon.$$

  Which is equivalent to saying that

$$f(B(x, \delta)) \subset B(f(x), \varepsilon).$$

- Sequential Continuity: $(x_n)_{n \geq 0} \subset X$,

$$x_n \to x \Rightarrow f(x_n) \to f(x).$$

- topological continuity: $\forall U \subset Y$ open, $f^{-1}(U)$ is open.

Proposition 1.29:

The three definitions of continuity are equivalent.

Proof. $1 \Rightarrow 2$: Assume $f : X \to Y$ is continuous in the epsilon-delta sense. Let $(x_n)_{n \geq 0} \subset X$ such that $x_n \to x$. Given $\varepsilon > 0$, by continuity, $\exists \delta > 0$ such that $f(B(x, \delta)) \subset B(f(x), \varepsilon)$. Since $x_n \to x$, $\exists N$ such that $x_n \in B(x, \delta)$ for all $n \geq N$. Thus $f(x_n) \in B(f(x), \varepsilon)$ for all $n \geq N$, which means that $f(x_n) \to f(x)$.

$2 \Rightarrow 3$: Assume f is not topologically continuous. Then $\exists U \subset Y$ open such that $f^{-1}(U)$ is not open. Hence $\exists x \in f^{-1}(U)$ and a sequence $(x_n)_{n \geq 0} \subset X \setminus f^{-1}(U)$ such that $x_n \to x$. Then $f(x) \in U$ but $f(x_n) \notin U$ for all n. But U is open so $f(x_n) \not\to f(x)$, contradicting sequential continuity.

Lec 4

$3 \Rightarrow 1$: Let $x \in X$ and $\varepsilon > 0$. The preimage $f^{-1}(B(f(x), \varepsilon))$ contains the point x and is open as $B(f(x), \varepsilon)$ is open and f is topologically continuous. Thus, $\exists \delta > 0$ such that $B(x, \delta) \subset f^{-1}(B(f(x), \varepsilon))$. Hence, f is $\varepsilon - \delta$ continuous. □

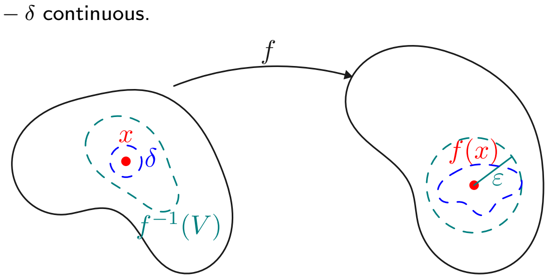

Figure 5: Topological continuity implies $\varepsilon - \delta$ continuity

Definition 1.30: Uniform and Lipschitz Continuity

Let (X, d_x) and (Y, d_Y) be metric spaces. We say that $f : X \to Y$ is UNIFORMLY CONTINUOUS if $\forall \varepsilon > 0, \exists \delta > 0$ such that

$$d_X(x, x') < \delta \Rightarrow d_Y(f(x), f(x')) < \varepsilon.$$

We say that f is LIPSCHITZ CONTINUOUS if $\exists L > 0$ such that

$$d_Y(f(x), f(x')) \leq L d_X(x, x') \forall x, x' \in X.$$

The constant L is called the LIPSCHITZ CONSTANT of f.

Example 1.31:

Fix $x_0 \in X$ and define $f := d(x, x_0)$. Then f is 1-Lipschitz.

Proof. Notice that $Y = [0, \infty)$ with the Euclidean metric. Then, for all $x, y \in X$ we have

$$d(f(x), f(y)) = |f(x) - f(y)| = |d(x, x_0) - d(y, x_0)| \leq d(x, y).$$

□

Theorem 1.32: Banach Fixed Point Theorem

Let (X, d) be a complete metric space and $T : X \to X$ λ-Lipschitz with $\lambda \in (0, 1)$ (sometimes called a CONTRACTION). Then, T has a unique FIXED POINT ($\exists! x \in X$ such that $T(x) = x$).

Proof. Fix $x_0 \in X$ and define $x_1 = T(x_0), x_2 = T(x_1), \ldots, x_n = T(x_{n-1}), \ldots$. We will show that $(x_n)_{n \geq 0}$ is a Cauchy sequence, and thus converges to some $x \in X$.

Since T is a contraction,

$$\begin{aligned} d(x_{n+1}, x_n) &= d(T(x_n), T(x_{n-1})) \\ &\leq \lambda \cdot d(x_n, x_{n-1}) \leq \ldots \\ &\leq \lambda^n \cdot d(x_1, x_0). \end{aligned}$$

Since the distance is symmetric, w.l.o.g assume $m < n$. Then, by triangle inequality,

$$\begin{aligned} d(x_n, x_m) &\leq \sum_{k=m}^{n-1} d(x_{k+1}, x_k) \\ &\leq \sum_{k=m}^{n-1} \lambda^k \cdot d(x_1, x_0) \\ &\leq d(x_1, x_0) \cdot \lambda^m \cdot \frac{1}{1 - \lambda}. \end{aligned}$$

All terms besides λ^m are constants, and $\lambda^m \to 0$ as $m \to \infty$, thus $(x_n)_{n \geq 0}$ is a Cauchy sequence.

Since X is complete, $\exists x \in X$ such that $x = \lim_{n \to \infty} x_n$. Since T is continuous,

$$T(x) = \lim_{n \to \infty} T(x_n) = \lim_{n \to \infty} x_{n+1} = x.$$

So x is indeed a fixed point.

For uniqueness, suppose x, y are two fixed points. Then,

$$d(x, y) = d(T(x), T(y)) \leq \lambda \cdot d(x, y) < d(x, y).$$

□

We now like to extend our definition for compactness from Analysis I to metric spaces. To that extent, we need the following definition.

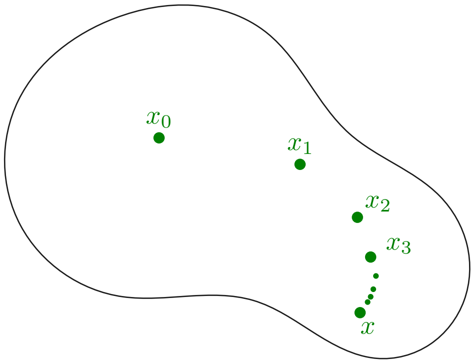

Figure 6: Proof of the Banach Fixed Point Theorem

Definition 1.33: Cover

Given a set X and $E \subset X$, we say that $\mathcal{U} = \{U_i, i \in I\}$ is a COVER of E if $E \subset \bigcup \mathcal{U} = \bigcup_{i \in I} U_i$.

If $V \subset \mathcal{U}$ is still a cover of E, we call V a SUBCOVER.

If $\mathcal{U}$ is a collection of open sets, we call it an OPEN COVER.

Definition 1.34: Compactness

A set $K \subset X$ is called

(1) SEQUENTIALLY COMPACT if for every sequence $(x_n)_{n \geq 0} \subset K$, there exists a convergent subsequence $(x_{n_k})_{k \geq 0}$ such that $x_{n_k} \to x \in K$.

(2) TOPOLOGICALLY COMPACT if for every open cover $\mathcal{U}$ of K, there exists a finite subcover $V \subset \mathcal{U}$ of K.

Example 1.35: Analysis I

Bolzano-Weierstrass: Any closed interval $[a, b] \subset \mathbb{R}$ is sequentially compact.

$[0, 1] \cap \mathbb{Q}$ is not sequentially compact. It is also not topologically compact, as $\mathbb{Q} \subset \{x_n | n \geq 0\}$ and we consider the open balls $\mathcal{U} = \{B(x_n, 2^{-n-10^3})\}$ then the total length of the intervals is $4 \cdot 2^{-10^3}$, which is less than 1, thus we cannot cover $[0, 1] \cap \mathbb{Q}$ with a finite number of intervals.

Proposition 1.36:

The two definitions of compactness are equivalent.

We will now show that (1) $\Rightarrow$ (2), and we will show the converse later.

Proof. Assume $K \subset X$ is sequentially compact. Let $\{U_i\}$ be an open cover. Thus $\forall x \in K, \exists U_i$ open, such that $x \in U_i$. Given $x \in K$, let

$$r(x) = \min\{\sup\{r > 0 : B(x, r) \subset U_i \in \mathcal{U}\}, 1\}.$$

Given $x \in K$, select U_i, such that $B\left(x, \frac{r(x)}{2}\right) \subset U_{i(x)}$[1]

Pick any $x_0 \in K$ and define

$$\mathcal{V} := \{\underbrace{U_i(x_0)}_{=U_0}, \underbrace{U_i(x_1)}_{=U_1}, \dots\},$$

where $x_1 \in K \setminus U_0$, $x_2 \in K \setminus (U_0 \cup U_1)$, and so on.

[1] We select $\frac{r}{2}$ in case the Supremum is not attained.

Doing so, unless I find a finite subcover, I will produce a sequence $x_n \in K \setminus \bigcup_{k=0}^{n-1} U_{i(x_k)}$. By sequential compactness, this sequence has a subsequence x_{n_l} such that $x_{n_l} \to x \in K$ with $r(x) > 0$. By construction $r(x_n) \to 0$! But $B_{\frac{r(x)}{2}} \subset U_{i(x)}$. Thus, for l large enough, $x_{n_l} \in U_{i(x)}$, contradicting the construction of the sequence. □ Lec 5

For the converse, the proof is a bit shorter.

Proof. Given $(x_n)_{n \geq 0} \subset K$, we want to show that there exists a subsequence $(x_{n_k})_{k \geq 0}$ such that $x_{n_k} \to x \in K$.

Assume by contradiction, $\forall x \in K$, x is not an accumulation point of $(x_n)_{n \geq 0}$ so $\forall x \in K, \exists \epsilon(x) > 0$ such that (x_n) visits $B(x, \epsilon(x))$ only finitely many times. Thus, $x_n \in K \setminus B(x, \varepsilon(x)) \forall n \geq N(x)$.

Define now
$$\mathcal{U} = \{B(x, \varepsilon(x)) | x \in K\}.$$
Since K is topologically compact,

$$K \subset \bigcup_{i=1}^{N} B(x_i, \varepsilon(x_i)).$$

This would imply that our sequence only has finitely many terms, contradicting the fact that it is a sequence. □

Corollary 1.37:

If $K \subset X$ is compact, then

1. K is closed,
2. K is complete,
3. If $A \subset X$ is closed, $K \cap A$ is compact.

Proposition 1.38:

If $f : X \to Y$ is continuous and $K \subset X$ is compact, then $f(K)$ is compact.

Proof. The goal is to show that $f(K)$ is topologically compact. Given $\mathcal{V}$ as an open cover of $f(K)$, we have that $\mathcal{U} = \{f^{-1}(V) | V \in \mathcal{V}\}$ is an open cover of K. Since K is compact, there exists a finite subcover $f^{-1}(V_1), \dots, f^{-1}(V_n)$ of K. Thus $V_1, \dots, V_n$ is a finite subcover of $f(K)$, and thus $f(K)$ is compact. □

Theorem 1.39:

Given $f : X \to \mathbb{R}$ continuous, $K \subset X$ compact, such that $\sup\{f(x) | x \in K\}$ and $\inf\{f(x) | x \in K\}$ are finite, then $\exists t \in K$ such that $f(t) = \sup\{f(x) | x \in K\}$

Proof. By definition of the supremum, $\exists (x_n)_{n \geq 0} \subset K$ such that $f(x_n) \to \sup\{f(x) | x \in K\}$ as $n \to \infty$. Since K is compact, $\exists (x_{n_k})_{k \geq 0}$ such that $x_{n_k} \to t \in K$. By continuity of f, $f(x_{n_k}) \to f(t)$ as $k \to \infty$. Thus, $f(t) = \sup\{f(x) | x \in K\}$. □

Since in this course, we will mostly work in $\mathbb{R}^n$, we will now show the following theorem.

Theorem 1.40: Heine-Borel

$K \subset \mathbb{R}^n$ is compact if and only if K is closed and bounded.

Note

We call a set $K \subset \mathbb{R}^n$ BOUNDED if $\exists M > 0$ such that

$$B(0, M) \supset K.$$

Proof. $\Rightarrow$: K is closed by corollary 1.37.

If K were unbounded, then $\exists (x_N)_{N\geq 0} \subset K$ such that $\|X_N\| \geq N \forall N$. Any subsequence of $(x_N)_{N\geq 0}$ is also unbounded. Indeed, by triangle inequality

$$x_{N_k} \to x \Leftrightarrow d(x_{N_k}, x) \to 0 \Rightarrow \|x_{N_k}\| < \|x\| + \|x_{N_k} - x\| \to \|x\|.$$

Thus, K is not sequentially compact, contradicting the fact that K is compact.

$\Leftarrow$: Our goal will be to show, that given $N \in \mathbb{N}$, $[-N, N]^n \subset \mathbb{R}^n$ is compact. This is sufficient, since if $K \subset \mathbb{R}^n$ is closed and bounded, then $K \subset B(0, N) \subset [-N, N]^n$ for some $N \in \mathbb{N}$, and thus K is a closed subset of a compact set, and thus compact by corollary 1.37.

We want to reduce the problem to Bolzano-Weierstrass in $\mathbb{R}$. Given $(x_k) \subset [-N, N]^n$, we can write $x_k = (x_{k,1}, \ldots, x_{k,n})$ where $x_{k,i} \in [-N, N]$.

1. Look at the sequence $(x_{k,1})_{k\geq 0} \subset [-N, N]$. By Bolzano-Weierstrass[2], there exists a increasing sequence $(k_m^{(1)})_{m\geq 0}$ in $\mathbb{N}$ such that

$$x_{k_m^{(1)},1} \to x_1 \in [-N, N].$$

2. Look at the sequence $(x_{k_m^{(1)},2})_{m\geq 0} \subset [-N, N]$. By Bolzano-Weierstrass, there exists a increasing sequence $(k_m^{(2)})_{m\geq 0}$ in $\mathbb{N}$ such that

$$x_{k_m^{(2)},2} \to x_2 \in [-N, N].$$

Since $(k_m^{(2)})_{m\geq 0}$ is a subsequence of $(k_m^{(1)})_{m\geq 0}$, we also have

$$x_{k_m^{(2)},1} \to x_1 \in [-N, N].$$

We continue this process, in each step making one new index converge, and keeping the previous ones converging. After n steps, we have an increasing sequence $(k_m^{(n)})_{m\geq 0}$ in $\mathbb{N}$ such that

$$x_{k_m^{(n)},i} \to x_i \in [-N, N] \quad \forall i = 1, \ldots, n.$$

Let $x = (x_1, \ldots, x_n)$, then by Lemma 1.17,

$$x_{k_m^{(n)}} \to x \in [-N, N]^n.$$

So K is sequentially compact. □

Tip 1.41:

The idea to split the proof into n times applying an argument from $\mathbb{R}$ is a very typical idea in Analysis II.

We now want to talk about CONNECTEDNESS. As a preliminary, in any metric space, $X, \varnothing$ are always open and closed (sometimes called clopen).

Question

Given $X = \mathcal{S}^2 = \{x \in \mathbb{R}^3 | \|x\| = 1\}$, equipped with the Euclidean metric restricted to $\mathcal{S}^2$.

Is it possible to write $X = U \cup V$ sich that $U \cap V = \varnothing$ and U, V are open in X and non-empty?

Definition 1.42: Connectedness

Given a metric space (X, d), we say that $A \subset X$ (nonempty) is DISSCONNECTED if $\exists U, V$ open, disjoint, such that $A \subset U \cup V$ and $A \cap U, A \cap V$ are non-empty.

We say that A is CONNECTED if it is not disconnected.

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In other words, disconnected means that we have an open cover of A with two disjoint sets.

[2] Any sequence in a compact interval has a convergent subsequence

Tip 1.43:

Working with connectedness is usually done by contraposition, since the definition is an existence statement.

Proposition 1.44: Connected subsets of $\mathbb{R}$

$E \subset \mathbb{R}$ is connected if and only if E is an interval.

Notice that E is an interval if and only if

$$\forall x < y \in E, [x, y] \subset E.$$

Proof. $\Rightarrow$: By contraposition, assume E is not an interval. Then, $\exists x < y \in E$ such that $[x, y] \not\subset E$. So $\exists z \in [x, y]$ such that $z \notin E$.

Define $U = (-\infty, z)$ and $V = (z, \infty)$. Then, U, V are open, disjoint, and $E \subset U \cup V$. Moreover, $E \cap U$ and $E \cap V$ are non-empty since $x \in E \cap U$ and $y \in E \cap V$. Thus, E is disconnected.

$\Leftarrow$: By contraposition, assume E is disconnected. Then, $\exists U, V$ open, disjoint, such that $E \subset U \cup V$ and $E \cap U, E \cap V$ are non-empty.

Thus, pick $x \in E \cap U$ and $y \in E \cap V$ such that $x < y$[3]. Consider the supremum of the following set:

$$t* = \sup\{t \geq x \mid [x, t] \subset U\}.$$

This supremum is well defined since $t* < y$ because $[x, y] \not\subset U$.

From this we also find that $t* \in \mathbb{R} \setminus V$ since V is open and disjoint from U. But also $t* \in \mathbb{R} \setminus U$ since if $t* \in U$, then by openness of U, $\exists \varepsilon > 0$ such that $B(t*, \varepsilon) \subset U$. Thus, $[x, t* + \varepsilon] \subset U$, contradicting the definition of $t*$.

So $t* \in \mathbb{R} \setminus (U \cup V) \subset \mathbb{R} \setminus E$, and thus $[x, y] \not\subset E$. So E is not an interval. □

Proposition 1.45:

Given $(X, d_X), (Y, d_Y)$ metric spaces, $f : X \to Y$ continuous and $E \subset X$ connected, then $f(E)$ is connected.

Proof. By contraposition, assume $f(E)$ is disconnected. Then, $\exists V_1, V_2$ open, disjoint, such that $f(E) \subset V_1 \cup V_2$ and $f(E) \cap V_1, f(E) \cap V_2$ are non-empty.

Define now $U_i = f^{-1}(V_i)$, which are open since f is continuous. Then, U_1, U_2 are open, disjoint, and $E \subset U_1 \cup U_2$. Also, they are non-empty since $f(E) \cap V_i$ are non-empty. Thus, E is disconnected. □

Corollary 1.46: Intermediate Value Theorem

Given a connected metric space (X, d) and $f : X \to \mathbb{R}$ continuous, such that $f(x) = a \leq f(y) = b$. Then $\exists t \in X$ such that $f(t) = c$ for all $c \in [a, b]$.

Proof. Skipped...

Since X is connected, $f(X)$ is connected, and thus an interval. Since $a, b \in f(X)$, we have $[a, b] \subset f(X)$, and thus $\exists t \in X$ such that $f(t) = c$. □

Definition 1.47: Curve

Given a metric space (X, d), a CURVE or PATH in X is a map $\gamma : [0, 1] \to X$ which is continuous.

$\gamma(0)$ is called the STARTING POINT of γ, and $\gamma(1)$ is called the ENDPOINT of γ.

γ is called CLOSED or LOOP if $\gamma(0) = \gamma(1)$.

[3] Can be assumed w.l.o.g.

Definition 1.48: Path Connectedness

Given a metric space (X, d), we call $E \subset X$ path connected if $\forall x, y \in E, \exists$ a path joining x and y, i.e. $\exists \gamma : [0,1] \to E$ continuous such that $\gamma(0) = x$ and $\gamma(1) = y$.

Proposition 1.49:

Given (X, d), E path connected, then E is connected.

Proof. Assume by contraposition that E is disconnected. Then, $\exists U_1, U_2$ open, disjoint, such that $E \subset U_1 \cup U_2$ and $\exists x_i \in E \cap U_i$.

Assume by contradiction, $\exists \gamma : [0,1] \to E$ continuous such that $\gamma(0) = x_1$ and $\gamma(1) = x_2$. A let $V_i = \gamma^{-1}(U_i)$, which are open since γ is continuous and disjoint. So $[0,1]$ is disconnected, contradicting the fact that $[0,1]$ is connected. □

The converse is not true, as the following example shows.

Example 1.50: Topologist's Sine Curve

Consider $(\mathbb{R}^2, d_{\text{Eucl}})$ and the set

$$E = \{0\} \times [-1,1] \cup \{(t, \sin(\frac{1}{t}))|t > 0\}.$$

This set is connected. However it is not path connected since intuitively, if we want to connect $(0,0)$ to $(1, \sin(1))$, we need to go through infinitely many oscillations of the sine curve, which is not possible with a continuous path.

Theorem 1.51:

In $(\mathbb{R}^n, d_{\text{Eucl}})$, given $U \subset \mathbb{R}^n$ open. U is connected if and only if U is path connected.

We first like to define the composition of paths. Given $\gamma_1 : [0,1] \to X$ and $\gamma_2 : [0,1] \to X$ such that $\gamma_1(0) = \gamma_2(0)$, we can define $\gamma_1^* = \gamma_1(1-t)$ as the reverse of γ_1. Then, the composition is defined as

$$\gamma_3(t) = \begin{cases} \gamma_1^*(2t) & t \in [0, \frac{1}{2}] \\ \gamma_2(2t-1) & t \in [\frac{1}{2}, 1] \end{cases}.$$

Proof. $\Leftarrow$: By Proposition 1.49.

$\Rightarrow$: We want to show that we can pick $x_0 \in U$ and join it with any other point $x \in U$ with a path. This is enough since if we can join x_0 to x and x_0 to y, then we can compose the paths to join x and y.

Define a set $G \subset U$ as

$$G := \{x \in U \mid \exists \text{ path } \gamma : [0,1] \to U : \gamma(0) = x_0, \gamma(1) = x\}.$$

We will now proof that G is open and that $U \setminus G$ is open, which since $x_0 \in G$ and U is connected, will imply that $G = U$.

The key observation is that for every $x \in U$, $\exists B_r(x) \subset U$ for some $r > 0$ since U is open. So $y \in B_r(x) \in G$ if and only if $x \in G$. This is because the map $t \in [0,1] \mapsto \gamma(t) = (1-t)x + ty$ is a path joining x and y.

This immediately implies that G is open, since if $x \in G$, then $B_r(x) \subset G$.

Moreover, if $x \in U \setminus G$, then $B_r(x) \subset U \setminus G$ since if $y \in B_r(x)$, then $x \in G$ if and only if $y \in G$. Thus, $U \setminus G$ is open. □

Proposition 1.52: Continuity on compact sets

Suppose $(X, d_X), (Y, d_Y)$ are metric spaces, $f : X \to Y$ is continuous, and $K \subset X$ is compact. Then, $f|_K : K \to Y$ is uniformly continuous.

Proof. Let $\varepsilon > 0$. By usual continuity of f, $\forall x \in K, \exists \delta_x > 0$ such that

$$f(B(x, \delta_x)) \subset B\left(f(x), \frac{\varepsilon}{2}\right).$$

Consider the open cover $\mathcal{U} = \{B(x, \frac{\delta_x}{2})|x \in K\}$ of K. Since K is compact, there exists a finite subcover $B(x_1, \frac{\delta_{x_1}}{2}), \dots, B(x_n, \frac{\delta_{x_n}}{2})$ of K. Let $\delta = \min\{\frac{\delta_{x_i}}{2}|i = 1, \dots, n\}$. We will show that this δ works for uniform continuity.

If $x \in K$ and $y \in B(x, \delta)$, then

$$x \in B(x_i, \frac{\delta_{x_i}}{2}) \text{ for some } i \in \{1, \dots, n\}.$$

So

$$d(x,y) < \delta \leq \frac{\delta_{x_i}}{2} \Rightarrow d(x_i, y) < \delta(x)..$$

But then

$$f(B(x, \delta)) \subset f(B(x_i, \delta_{x_i})) \subset B\left(f(x_i), \frac{\varepsilon}{2}\right) \subset B\left(f(x), \varepsilon\right).$$

□

1.2 Normed Vector Spaces

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Before starting with differentiating functions of several variables, we need to add some structure to our metric spaces.

Definition 1.53: Normed Vector Space over $\mathbb{R}$

Let V be a VS over $\mathbb{R}$. A map $\|\cdot\| : V \to [0, \infty)$ is called a NORM if $\forall v, u \in V$ and $\alpha \in \mathbb{R}$, the following hold:

1. Definite: $\|v\| = 0$ if and only if $v = 0$.
2. Homogeneous: $\|\alpha v\| = |\alpha| \cdot \|v\|$
3. Triangle inequality: $\|v + u\| \leq \|v\| + \|u\|$.

Example 1.54: Examples on $\mathbb{R}^n$

$\mathbb{R}^n$ with the Euclidean norm $\|x\|_{\text{Eucl}} = \sqrt{\sum_{i=1}^n x_i^2}$ is a normed vector space.

$\mathbb{R}^n$ with the p-norm $\|x\|_p = (\sum |x_i|^p)^{\frac{1}{p}}$ for $p \geq 1$ is a normed vector space.

$\mathbb{R}^n$ with the ∞-norm $\|x\|_\infty = \max |x_i|$ is a normed vector space.

From now on, we will write $|\cdot|$ instead of $\|\cdot\|$ when we use the Euclidean norm.

Definition 1.55: Hilbert-Schmidt norm

Let $M_{m \times n}(\mathbb{R})$ be the set of $m \times n$ matrices with real entries. We define the HILBERT-SCHMIDT NORM on $M_{m \times n}(\mathbb{R})$ as

$$\|M\|_2 = \sqrt{\text{tr}(M^T M)} = \sqrt{\sum_{i=1}^n |Me_i|^2}.$$

To see the last equality, we square the Hilbert-Schmidt norm, we get

$$\|M\|_2^2 = \operatorname{tr}(M^T M) = \sum_{i=1}^n \sum_{j=1}^m M_i^j M_i^j.$$

But also

$$[Me_i]^j = M_i^j \Rightarrow |Me_i|^2 = \sum_{j=1}^m M_i^j M_i^j.$$

Lemma 1.56:

For every $x \in \mathbb{R}^n$ and $M : \mathbb{R}^n \to \mathbb{R}^m$ linear,

$$|Mx| \leq \|M\|_2 \cdot |x|.$$

Proof. We can write $x = \sum_{i=1}^n x_i e_i$. Thus, we compute

$$\begin{aligned} |Mx|^2 &= \sum_{i=1}^m [(Mx)^j]^2 = |M(\sum_{i=1}^n x_i e_i)|^2 \\ &= \left|\sum_{i=1}^n x_i M e_i\right|^2 \\ &\leq \left(\sum_{i=1}^n |x_i M e_i|\right)^2 = \left(\sum_{i=1}^n |x_i||Me_i|\right)^2 \\ &\leq |x|^2 \cdot \left(\sum_{i=1}^n |Me_i|^2\right) = |x|^2 \cdot \|M\|_2^2. \end{aligned}$$

□

Example 1.57: Norms on function spaces

Given $V = \{f : [a,b] \to \mathbb{R} \mid f \text{ continuous}\}$ we have the L^p-NORM for $p \geq 1$

$$\|f\|_{L^p} := \left(\int_a^b |f(t)|^p dt\right)^{\frac{1}{p}}.$$

And the SUPREMUM NORM

$$\|f\|_{L^\infty} := \sup_{t \in [a,b]} |f(t)|.$$

Proposition 1.58: Norm implies metric

Every normed vector space is a metric space with the metric defined as

$$d(x,y) = \|x - y\|.$$

Proof. d is definite because $\|x - y\| = 0$ if and only if $x - y = 0$ if and only if $x = y$.

d is symmetric because $\|x - y\| = \|(-1)(y - x)\| = \|y - x\|$.

d satisfies the triangle inequality, because

$$\|x - z\| = \|(x - y) + (y - z)\| \leq \|x - y\| + \|y - z\|.$$

□

Definition 1.59: Inner Product Vector Space

Let V be a VS over $\mathbb{R}$. A INNER PRODUCT is a map $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ such that $\forall u, v, w \in V$ and $\alpha \in \mathbb{R}$:

1. Symmetry: $\langle u, v \rangle = \langle v, u \rangle$.
2. Bilinearity: $\langle \alpha u + \beta v, w \rangle = \alpha \langle u, w \rangle + \beta \langle v, w \rangle$.
3. Definite: $\langle v, v \rangle \geq 0$ and $\langle v, v \rangle = 0 \Leftrightarrow v = 0$.

Lemma 1.60:

Let V be an inner product vector space. Then, $\|\cdot\| : V \to [0, \infty)$ defined as

$$\|v\| = \sqrt{\langle v, v \rangle}.$$

is a norm on V. So every inner product vector space is a normed vector space.

Proof. Exercise Sheet and Linear Algebra: Uses Cauchy-Schwarz inequality,

$$\langle v, w \rangle \leq \sqrt{\langle v, v \rangle} \cdot \sqrt{\langle w, w \rangle}.$$

□

Going back to the beginning of the lecture, we now know that $\mathbb{R}^n$ is an inner product vector space with the inner product defined as $\langle x, y \rangle = \sum_{i=1}^n x_i y_i$, and thus a normed vector space and a metric space.

From now on, we will by default work with the Euclidean norm and metric on $\mathbb{R}^n$. The most central definitions are the limit of a function on an open set $U \subset \mathbb{R}^n$

$$y = \lim_{x \to x_0} f(x) \Leftrightarrow \forall (x_k) \subset U \text{with } x_k \to x_0, f(x_k) \to y.$$

and the continuity of a function

$$\lim_{x \to x_0} f(x) = f(x_0).$$

One useful trick to compute a limit of a function $\mathbb{R}^2 \to \mathbb{R}$ as $r \to 0$ is to use polar coordinates. We can write $x = (r\cos\theta, r\sin\theta)$, and thus for example

$$\lim_{\to 0} \frac{xy}{\sqrt{x^2 + y^2}} = \lim_{r \to 0} \frac{r^2 \cos\theta \sin\theta}{r} = \lim_{r \to 0} r\cos\theta\sin\theta = 0.$$

Lec 8

2 Multidimensional Differentiation

We start with the most important definition of the course.

Definition 2.1: Differential

Let $U \subset \mathbb{R}^n$ open, $f : U \to \mathbb{R}^m$. We say that f is DIFFERENTIABLE at $x_0 \in U$ if there exists a linear map $L : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{x\to 0} \frac{f(x_0+x) - f(x_0) - L(x)}{|x|} = 0.$$

L is called the DIFFERENTIAL of f at x_0, and is denoted by Df_{x_0} or $Df(x_0)$.

Claim 2.2:

If $n = m = 1$, then $L(x) = f'(x_0)x$ where $f'(x_0)$ is the usual derivative of f at x_0.

Solution. We know $f'(x_0) = \lim_{x\to 0} \frac{f(x_0+x)-f(x_0)}{x}$, so we can write

$$\begin{aligned}
&\lim_{x\to 0} \frac{f(x_0+x) - f(x_0) - f'(x_0)x}{|x|} \\
&= \lim_{x\to 0} \frac{f(x_0+x) - f(x_0)}{x} - f'(x_0) \\
&= f'(x_0) - f'(x_0) = 0.
\end{aligned}$$

For some intuition consider the following examples.

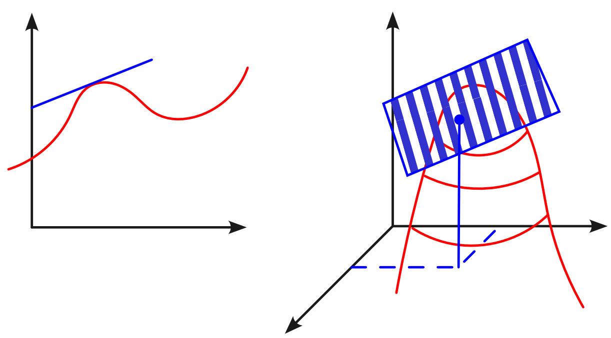

Figure 7: Differential of a function $\mathbb{R} \to \mathbb{R}$ and of a function $\mathbb{R}^2 \to \mathbb{R}$.

The equation for the slope in the first case would be

$$y = f(x_0) + f'(x_0)(x - x_0).$$

Whereas the equation for the tangent plane in the second case would be

$$y = f(x_0) + Df_{x_0}(x - x_0).$$

Lemma 2.3: Differential Componentwise

Let $U \subset \mathbb{R}^n$ open, $f : U \to \mathbb{R}^m$ and $x_0 \in U$. Then f is differentiable at x_0 if and only if $f_i : U \to \mathbb{R}$ is differentiable at x_0 $\forall i = 1, \dots, m$, where f_i is the i-th component of f. Moreover,

$$Df_{x_0}(x) = \begin{pmatrix} Df_{1,x_0}(x) \\ \vdots \\ Df_{m,x_0}(x) \end{pmatrix}.$$

This lemma allows us to reduce proofs to the case $m = 1$, which is usually easier to work with.

Proof. Recall the definition of the derivative of f at x_0:

$$\lim_{x\to 0} \frac{f(x_0+x) - f(x_0) - L(x)}{|x|} = 0.$$

This can converge if and only if $\forall i = 1, \dots, m$,

$$\lim_{x\to 0} \frac{f_i(x_0+x) - f_i(x_0) - L_i(x)}{|x|} = 0.$$

But this is equivalent to f_i being differentiable at x_0 with differential L_i for all $i = 1, \dots, m$. □

A convenient notation is the following

Definition 2.4: Big and Little O

Given $f, g : U \subset \mathbb{R}^n \to \mathbb{R}^m$ and $x_0 \in U$, we say that

$$f(x) = O(g(x)) \text{ as } x \to x_0 \Leftrightarrow \lim_{x\to 0} \frac{|f(x)|}{|g(x)|} < \infty.$$

Similarly, we say that

$$f(x) = o(g(x)) \text{ as } x \to x_0 \Leftrightarrow \lim_{x\to 0} \frac{|f(x)|}{|g(x)|} = 0.$$

With this notation, f is differentiable at x_0 if and only if

$$R(x) = f(x_0+x) - f(x_0) - Df_{x_0}(x) = o(|x|).$$

Definition 2.5: Directional derivatives

Let $U \subset \mathbb{R}^n$ open, $f : U \to \mathbb{R}^m$ and $x_0 \in U$. Given $v \in \mathbb{R}^n$, we define the DIRECTIONAL DERIVATIVE of f at x_0 along v as

$$\partial_v f(x_0) = \lim_{t\to 0} \frac{f(x_0+tv) - f(x_0)}{t},$$

provided the limit exists.

Proposition 2.6:

Given $U \subset \mathbb{R}^n$ open, $f : U \to \mathbb{R}^m$ and $x_0 \in U$. f is differentiable at $x_0 \in U$ implies that $\partial_v f(x_0)$ exists for all $v \in \mathbb{R}^n$, and

$$\partial_v f(x_0) = Df_{x_0}(v).$$

Proof. We can write

$$f(x_0+x) - f(x_0) - Df_{x_0}(x) = o(|x|) \text{ as } x \to 0.$$

Thus for any sequence going to 0, we have that

$$\frac{f(x_0+x) - f(x_0) - Df_{x_0}(x)}{|x|} \to 0.$$

In particular, for the sequence $x = tv$ with $t \to 0$, we have

$$\frac{f(x_0+tv) - f(x_0) - Df_{x_0}(tv)}{|tv|} \to 0.$$

Since Df_{x_0} is linear, we can write $Df_{x_0}(tv) = tDf_{x_0}(v)$, and since $|tv| = |t||v|$, with $|v|$ finite, we can write

$$\underbrace{\frac{f(x_0+tv) - f(x_0)}{t}}_{=\partial_v f(x_0)} - Df_{x_0}(v) \to 0.$$

□

Definition 2.7: Partial derivatives

Let $U \subset \mathbb{R}^n$ open, $f : U \to \mathbb{R}^m$ and $x_0 \in U$. Given $i \in \{1, \dots, n\}$, we define the PARTIAL DERIVATIVE of f at x_0 along the i-th coordinate as

$$\partial_{e_i} f(x_0) =: \frac{\partial f}{\partial x_i}(x_0) = \partial_i f(x_0).$$

Notice that the partial derivatives are just the directional derivatives along the canonical basis vectors.

Example 2.8:

Let $n = 3, m = 1$ and $f(x_1, x_2, x_3) = \exp(x_2)[1 + x_1 x_3]$. Then, compute the partial derivative $\partial_1 f(0)$ and all partial derivatives at an arbitrary point $x = (x_1, x_2, x_3)$.

Solution. By definition

$$\begin{aligned}\partial_1 f(0,0,0) &= \lim_{t\to 0} \frac{f(0+t,0,0) - f(0,0,0)}{t} \\ &= \lim_{t\to 0} \frac{\exp(0)[1 + t\cdot 0] - \exp(0)[1 + 0 \cdot 0]}{t} \\ &= \lim_{t\to 0} \frac{1-1}{t} = 0.\end{aligned}$$

For the general case, we can define $g(t) = f(x_1 + t, x_2, x_3)$, so that $\partial_1 f(x) = g'(0)$. We can compute

$$\begin{aligned}\partial_1 f(x_1, x_2, x_3) &= \exp(x_2)x_3 \\ \partial_2 f(x_1, x_2, x_3) &= \exp(x_2)(1 + x_1 x_3) \\ \partial_3 f(x_1, x_2, x_3) &= \exp(x_2)x_1.\end{aligned}$$

In principle, the partial derivative can be calculated by fixing all the coordinates except the one we want to differentiate with respect to, and then applying the usual derivative in one variable.

The trick can be extended to a directional derivative along an arbitrary direction v

$$\partial_v f(x_0) = g'(0) \text{ where } g(t) = f(x_0 + tv).$$

Given $U \subset \mathbb{R}^n$ open, $f : U \to \mathbb{R}^m$. When $\partial_i f(x_0)$ exists for all $x_0 \in U$ then we define the function

$$\partial_i f : U \to \mathbb{R}^m, x \mapsto \partial_i f(x).$$

Theorem 2.9: Sufficient condition for differentiability

Let $U \subset \mathbb{R}^n$ open, $f : U \to \mathbb{R}^m$. If $\partial_i f(x)$ exists and is continuous for every $i = 1, \dots, n$, and every $x \in U$, then f is differentiable at every $x \in U$.

Moreover,

$$Df_{x_0}(x) = \big(\partial_1 f(x_0), \dots, \partial_n f(x_0)\big)\, x.$$

More explicitly, if we write $x = (x_1, \dots, x_n)$, then

$$Df_{x_0}(x) = \begin{pmatrix} \partial_1 f_1(x_0) & \dots & \partial_n f_1(x_0) \\ \vdots & \ddots & \vdots \\ \partial_1 f_m(x_0) & \dots & \partial_n f_m(x_0) \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

Proof. Fix $x_0 \in U$ and take $\varepsilon > 0$ such that $\{x \mid |x_i - x_{0,i}| < \varepsilon\} \subset U$. Take $x \in \mathbb{R}^n$ and define $x^{(k)} = x_0 + \sum_{i=1}^k x_i e_i$, so that $x^{(0)} = x_0$ and $x^{(n)} = x_0 + x$. Then, we can write

$$\begin{aligned} f(x_0 + x) - f(x_0) &= \sum_{k=1}^n f(x^{(k)}) - f(x^{(k-1)}) \\ &= \sum_{k=1}^n \partial_k f(y^{(k)}) x_k. \end{aligned}$$

We want to show the last equality by using the mean value theorem on a function $g_k(t) = f(x^{(k-1)} + te_k)$, so that $g_k(0) = f(x^{(k-1)})$ and $g_k(x_k) = f(x^{(k)})$.

By Lemma 2.3, we can assume $m = 1$ and thus $g_k : \mathbb{R} \to \mathbb{R}$, so that by the mean value theorem, $\exists \xi_k \in [x^{(k-1)}, x^{(k)}]$ such that

$$g_k(x_k) - g_k(0) = g_k'(\xi_k)x_k = \partial_k f(x^{(k-1)} + \xi_k e_k)x_k.$$

Letting $y^{(k)} = x^{(k-1)} + \xi_k e_k$, we get the desired equality.

From this, we can write

$$f(x_0 + x) - f(x_0) = \sum_{k=1}^n \partial_k f(y^{(k)}) x_k$$

.

But $\sum_{i=1}^{k-1} x_i e_i \to 0$ as $x \to 0$, and since $\xi_k \in [0, x_k]$, we have $\xi_k e_k \to 0$ as $x \to 0$. Thus, $y^{(k)} \to x_0$ as $x \to 0$.

$$\begin{aligned} f(x_0 + x) - f(x_0) &= \sum_{k=1}^n \partial_k f(y^{(k)}) x_k \\ &= \sum_{k=1}^n (\partial_k f(x_0) x_k + o(1)) \cdot |x| \end{aligned}$$

.

Thus, we can write

$$f(x_0 + x) - f(x_0) = \sum_{k=1}^n \partial_k f(x_0) x_k + o(|x|).$$

Which means that f is differentiable at x_0 with differential

$$Df_{x_0}(x) = \big(\partial_1 f(x_0), \dots, \partial_n f(x_0)\big)\, x.$$

□

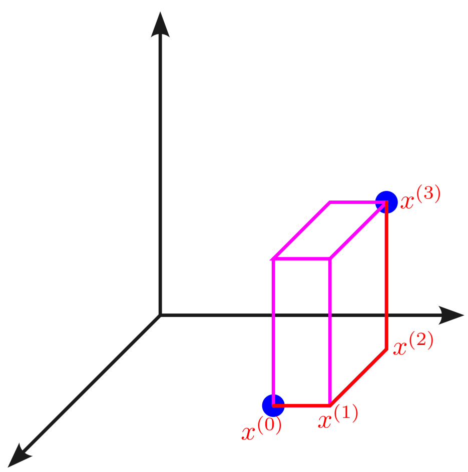

Figure 8: Definition of the $x^{(i)}$s.

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Definition 2.10: C^1 functions

Given $U \subset \mathbb{R}^n$ open, we define $C^1(U, \mathbb{R}^m)$ as

$$C^1(U, \mathbb{R}^m) := \{f : U \to \mathbb{R}^m : f \text{ cont. diff. in } U\}.$$

When $m = 1$, we write $C^1(U)$.

Definition 2.11: Jacobi Matrix

Given $U \subset \mathbb{R}^n$ open, $f : U \to \mathbb{R}^m$, differentiable at x_0. We define the JACOBI MATRIX of f at x_0 as

$$Df(x_0) = Jf(x_0) = (\partial_1 f(x_0), \dots, \partial_n f(x_0)).$$

Writing out the Jacobi matrix explicitly, we have

$$Jf(x_0) = \begin{pmatrix} \partial_1 f_1(x_0) & \dots & \partial_n f_1(x_0) \\ \vdots & \ddots & \vdots \\ \partial_1 f_m(x_0) & \dots & \partial_n f_m(x_0) \end{pmatrix}.$$

We observe, that if we take a vector $v = (v_1, \dots, v_n)^T = \sum_{i=1}^n v_i e_i$, then

$$Df_{x_0}(v) = \sum_{i=1}^n v_i Df_{x_0}(e_i) = \sum_{i=1}^n v_i \frac{\partial f}{\partial x_i} = Jf(x_0)v.$$

That is, $Jf(x_0)$ is the matrix of Df_{x_0} with respect to the canonical basis.

Theorem 2.12: Chain Rule

Given $U \subset \mathbb{R}^n$ open, $V \subset \mathbb{R}^m$ open and $f : U \to V$ and $g : V \to \mathbb{R}^k$. Assume that f is differentiable at $x_0 \in U$ and g is differentiable at $f(x_0) \in V$. Then, $g \circ f : U \to \mathbb{R}^k$ is differentiable at x_0, and

$$D(g \circ f)_{x_0} = Dg_{f(x_0)} \circ Df_{x_0}.$$

The idea of the proof is that

$$f(x_0 + x) - f(x_0) \approx L(x) \text{ and } g(y_0 + y) - g(y_0) \approx M(y).$$

Using $y = f(x_0 + x) - f(x_0)$, we can write

$$\begin{aligned} g(y_0 + f(x_0 + x)) - g(y_0) &\approx M(f(x_0 + x) - f(x_0)) \\ g(f(x_0 + x)) - g(f(x_0)) &\approx M(L(x)). \end{aligned}$$

Proof. We can write

$$\begin{aligned} f(x_0 + x) - f(x_0) &= L(x) + o(|x|) \\ g(y_0 + y) - g(y_0) &= M(y) + o(|y|). \end{aligned}$$

Then substituting $y = f(x_0 + x) - f(x_0)$, we get

$$\begin{aligned} &g(f(x_0 + x)) - g(f(x_0)) \\ &= g(y_0 + f(x_0 + x) - f(x_0)) - g(y_0) \\ &= M(f(x_0 + x) - f(x_0)) + o(|f(x_0 + x) - f(x_0)|) \\ &= M(L(x) + o(|x|)) + o(|L(x) + o(|x|)|) \\ &= M(L(x)) + o(|x|). \end{aligned}$$

Where we used the fact, that M and L are linear, so that $M(L(x))$ is linear in x, and thus $o(|L(x)|) = o(|x|)$, and $o(o(|x|)) = o(|x|)$. □

As a practical consequence of this,

$$J(g \circ f)(x_0) = Jg(f(x_0)) \cdot Jf(x_0).$$

Component-wise this means that

$$\frac{\partial (g \circ f)_k}{\partial x_i}(x) = \sum_{j=1}^m \frac{\partial g_k}{\partial y_j}(f(x)) \cdot \frac{\partial f_j}{\partial x_i}(x).$$

Theorem 2.13: Generalized Mean Value Theorem

Given $U \subset \mathbb{R}^n$ open, $f \in C^1(U)$. Assume $x_0 \in U$ and $h \in \mathbb{R}^n$ such that $x_0 + th \in U \, \forall t \in [0, 1]$. Then $f(x_0 + h) - f(x_0) = Df_{x_0 + \theta h}(h)$ for some $\theta \in [0, 1]$.

Proof. For $t \in [0, 1]$, define $g(t) = f(x_0 + th)$. Then,

$$f(x_0 + h) - f(x_0) = g(1) - g(0) \overset{\text{MVT}}{=} g'(\theta).$$

Since $g(t) = f \circ \gamma(t)$ where $\gamma(t) = x_0 + th$, by the chain rule,

$$g'(t) = Df_{\gamma(t)} \circ D\gamma_t = Df_{x_0 + th}(h).$$

Letting $t = \theta$, we get the desired result. □

Definition 2.14: Convex Sets

$A \subset \mathbb{R}^n$ is called CONVEX if $\forall x, y \in A$ and $\forall t \in [0, 1]$, we have $tx + (1 - t)y \in A$.

Exercise 2.15:

In $\mathbb{R}^n$, $B_r(x_0)$ and $\overline{B_r(x_0)}$ are convex.

Definition 2.16: Gradient

Given $U \subset \mathbb{R}^n$ open, $f \in C^1(U)$. Define the GRADIENT of f at x as the vector

$$\nabla f(x) = Jf(x)^T = \begin{pmatrix} \partial_1 f(x) \\ \vdots \\ \partial_n f(x) \end{pmatrix}.$$

Proposition 2.17:

Given $U \subset \mathbb{R}^n$ open, $f \in C^1(U)$. Assume U convex and that $\sup_{x \in U} |\nabla f(x)| \le M$ for some $M > 0$. Then, $\forall x, y \in U$,

$$|f(x) - f(y)| \le M|x - y|.$$

Proof. Let $x = y + h, y = x_0$. Then

$$\begin{aligned} |f(x) - f(y)| &= |f(x_0 + h) - f(x_0)| \\ &\overset{\text{MVT}}{=} |Df_\xi(h)| \\ &= |Jf(\xi)h| \\ &= |\nabla f(\xi)h| \\ &\overset{CS}{\le} |\nabla f(\xi)| \cdot |h| \\ &\le M|x - y|. \end{aligned}$$

We can apply the mean value theorem since U is convex, and thus contains the straight line between x and y. □

From this we can proof a similar result for $m \ge 2$.

Theorem 2.18: Differentiabilty vs Lipschitz

Given $U \subset \mathbb{R}^n$ open, $f \in C^1(U, \mathbb{R}^m)$. Suppose, that

$$\Lambda := \sup_{x \in U} \|Jf(x)\|_2 < \infty.$$

Then, f is Lipschitz with constant $\sqrt{m}\Lambda$.

Proof. We apply the previous proposition to each component of f. For $i = 1, \dots, m$, we have

$$|f(x) - f(y)|^2 = \sum_{i=1}^m |f_i(x) - f_i(y)|^2 \le \sum_{i=1}^m \Lambda_i^2 |x - y|^2 = m\Lambda^2 |x - y|^2.$$

Where $\Lambda_i = \sup_{x \in U} |\nabla f_i(x)|$. Since $|\nabla f_i(x)| \le \|Jf(x)\|_2$, we have $\Lambda_i \le \Lambda$ for all $i = 1, \dots, m$. □

As a consequence, if U is any (not necessarily convex) open subset of $\mathbb{R}^n$, and $f \in C^1(U, \mathbb{R}^m)$ and $B_r(z) \subset U$ is a ball, then f is Lipschitz on $B_r(z)$ with a constant that may depend on z and r. We then call f locally Lipschitz. Indeed every C^1 function is locally Lipschitz.

Proof. $\overline{B_r(z)}$ is closed and bounded, so compact. So the continuous function $\|Jf(x)\|_2$ attains its maximum on $\overline{B_r(z)}$. So we can apply the above theorem to conclude that f is Lipschitz on $B_r(z)$ with constant $\sqrt{m}\max_{x\in\overline{B_r(z)}}\|Jf(x)\|_2$. □

Theorem 2.19:

Given $U \subset \mathbb{R}^n$ open, $f : U \to \mathbb{R}^m$, differentiable with $Df(x) = 0$ for all $x \in U$. Then, f is constant on each connected component of U. In particular, if U is connected, then f is constant.

Proof. It suffices to consider $m = 1$. Assume U is non-empty and choose $x_0 \in U$. Consider the subset

$$G = \{x \in U \mid f(x) = f(x_0)\} \subset U.$$

Since f is continuous, G is closed in U.

Since U is open, given $x \in G$, we find $\varepsilon > 0$ such that $B_\varepsilon(x) \subset U$. Since every point $y \in B_\varepsilon(x)$ can be connected to x by a line segment, we can apply the mean value theorem to conclude that $f(y) = f(x) = f(x_0)$, so $B_\varepsilon(x) \subset G$. Thus, G is open in U.

Since U is connected, and G and $U \setminus G$ are both open in U, we must have $G = U$, so f is constant on U.

Notice that $x_0 \in G$ so it cannot be empty. □

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2.1 Higher Order Derivatives

Similar to Analysis One, we can define higher order derivatives by iterating the definition of the derivative.

Definition 2.20: C^n functions

Given $U \subset \mathbb{R}^n$ open, we define

$$C^0(U, \mathbb{R}^m) = \{f : U \to \mathbb{R}^m : f \text{ continuous}\}.$$

$$C^k(U, \mathbb{R}^m) = \{f \in C^{k-1} \mid \partial_i f \in C^{k-1}\}.$$

If $f \in C^k(U, \mathbb{R}^m)$, we say that f is k-times continuously differentiable.

If $f \in C^k(U, \mathbb{R}^m)$ and $i_1, \dots, i_k \in \{1, \dots, n\}$, then we define

$$\partial_{i_1} \dots \partial_{i_k} f = \partial_{i_1}(\partial_{i_2} \dots \partial_{i_k} f).$$

Example 2.21:

Let $f(x_1, x_2, x_3) = x_1 x_3^2 \sin(x_2)$. Then, we can compute

$$\begin{aligned}\partial_2\partial_1\partial_2\partial_3(f) &= \partial_2\partial_1\partial_2(2x_1x_3\sin(x_2)) \\ &= \partial_2\partial_1(2x_1x_3\cos(x_2)) \\ &= \partial_2(2x_3\cos(x_2)) \\ &= -2x_3\sin(x_2).\end{aligned}$$

Proposition 2.22: C^k is a vector space

If $f, g \in C^k(U), U \subset \mathbb{R}^n$ open, $V \subset \mathbb{R}^m$ open. Let $h : V \to \mathbb{R}^n$ such that $h(V) \subset U$ and $h \in C^k(V, \mathbb{R}^n)$. Then

1. $f + g \in C^k(U)$ and $f \cdot g \in C^k(U)$.
2. $f \circ h \in C^k(V)$.

Proof. 1. We argue by induction over k.

Base Case ($k = 0$): Very Similar to Analysis I $\Rightarrow$ Exercise.

Inductive Step: Assume the claim holds for $k-1$. Then

$$\partial_i(f + g) = \partial_i f + \partial_i g \in C^{k-1}(U),$$

since $\partial_i f, \partial_i g \in C^{k-1}(U)$ and thus by the inductive hypothesis $\partial_i f + \partial_i g \in C^{k-1}(U)$. Also $f + g \in C^{k-1}(U)$, so $f + g \in C^k(U)$.

Similarly, we can write

$$\partial_i(f \cdot g) = \partial_i f \cdot g + f \cdot \partial_i g \in C^{k-1}(U),$$

since $C^k \subset C^{k-1}$.

2. We argue by induction over k.

Base Case ($k = 0$): Since A is open, $f^{-1}(A)$ is open implying $h^{-1}(f^{-1}(A))$ is open, so $f \circ h$ is continuous.

Inductive Step: Assume the claim holds for $k-1$. Then, by the chain rule,

$$\partial_j(f \circ h) = \sum_{i=1}^{n} (\partial_i f) \circ h \partial_j h_i.$$

All of these are by assumptions in $C^{k-1}(V)$. By the inductive hypothesis, and by 1. we have $\partial_j(f \circ h) \in C^{k-1}(V)$, and since $f \circ h \in C^{k-1}(V)$, we have $f \circ h \in C^k(V)$. □

Theorem 2.23: Schwarz's Theorem

Given $U \subset \mathbb{R}^n$ open, $f \in C^2(U, \mathbb{R}^m)$. Then $\forall i, j \in \{1, \dots, n\}$, we have $\partial_i\partial_j f = \partial_j\partial_i f$.

Proof. If $i = j$, there is nothing to prove. By symmetry, wlog $i < j$. It suffices to consider the case $n = 2$ with $i = 1$ and $j = 2$. For a general n fix i_1, i_2 and consider the function

$$\tilde{f}_b(x_1, x_2) = f(y_1, \dots, x_1, \dots, x_2, \dots, y_n),$$

where y_k is fixed for $k \notin \{i_1, i_2\}$. for $b \in 1, \dots, m$.

Thus, WLOG, $n = 2, m = 1$. For $x = (x_1, x_2) \in U$ and $h > 0$ sufficiently small, define

$$F(h) = f(x_1 + h, x_2 + h) - f(x_1 + h, x_2) - f(x_1, x_2 + h) + f(x_1, x_2).$$

Consider the differentiable function

$$\phi : [0, 1] \to \mathbb{R}, t \mapsto f(x_1 + th, x_2 + h) - f(x_1 + th, x_2).$$

Thus, by the mean value theorem, $\exists \xi_1 \in (0, 1)$ such that

$$F(h) = \phi(1) - \phi(0) = \phi'(\xi_1).$$

By the chain rule, we can write

$$F(h) = (\partial_1 f(x_1 + \xi_1 h, x_2 + h) - \partial_1 f(x_1 + \xi_1 h, x_2))\, h.$$

Define now the function

$$\psi : [0, 1] \to \mathbb{R}, t \mapsto \partial_1 f(x_1 + \xi_1 h, x_2 + th).$$

Then, by the one dimensional mean value theorem, $\exists \xi_2 \in (0, 1)$ such that

$$F(h) = \psi(1) - \psi(0) = \psi'(\xi_2) = \partial_2\partial_1 f(x_1 + \xi_1 h, x_2 + \xi_2 h)h^2.$$

Similarly, we can also define $\tilde{\phi}, \tilde{\psi}$ by swapping the roles of x_1 and x_2, and get

$$F(h) = \partial_1\partial_2 f(x_1 + \tilde{\xi}_1 h, x_2 + \tilde{\xi}_2 h)h^2,$$

for suitable $\tilde{\xi}_1, \tilde{\xi}_2 \in (0, 1)$. Dividing by h^2 and letting $h \to 0$, we get

$$\begin{aligned}\partial_2\partial_1 f(x_1 + \xi_1 h, x_2 + \xi_2 h) &= \partial_1\partial_2 f(x_1 + \tilde{\xi}_1 h, x_2 + \tilde{\xi}_2 h) \\ \partial_2\partial_1 f(x) &= \partial_1\partial_2 f(x).\end{aligned}$$

□

Corollary 2.24:

Given $U \subset \mathbb{R}^n$ open, $f \in C^k(U)$. Then $\forall i_1, \dots, i_k \in \{1, \dots, n\}$ and for every permutation $\sigma \in S_k$, we have

$$\partial_{i_1} \dots \partial_{i_k} f = \partial_{i_{\sigma(1)}} \dots \partial_{i_{\sigma(k)}} f.$$

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A useful shorthand is the following.

Definition 2.25: Multi-index

A MULTI-INDEX is a tuple $\alpha = (\alpha_1, \dots, \alpha_n) \subset \mathbb{N}^n$. We define $|\alpha| = \alpha_1 + \dots + \alpha_n$ as the length of α. We say that $\beta \leq \alpha$ if $\beta_i \leq \alpha_i$ for all $i = 1, \dots, n$. We define the factorial

$$\alpha! = \alpha_1! \cdots \alpha_n!.$$

With this notation we can for example write a polynomial of degree k in n variables compactly as

$$P(x) = \sum_{|\alpha| \leq k} c_\alpha x^\alpha.$$

With this notation, we can also neatly write multiple derivatives as

$$\partial^\alpha f = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} f,$$

where $f \in C^k$ and $|\alpha| \leq k$.

Theorem 2.26: Taylor's Theorem

Given $U \subset \mathbb{R}^n$ open, $f \in C^{k+1}(U), k \geq 0$. Let $x_0 \in U$ and $h \in \mathbb{R}^n$ such that $x_0 + th \in U \, \forall t \in [0,1]$. Then

$$f(x_0 + h) = \sum_{|\alpha| \leq k} \partial^\alpha f(x_0) \frac{h^\alpha}{\alpha!} + R_{k+1} f(x_0, h).$$

Where the remainder term $R_{k+1} f(x_0, h)$ is given by

$$R_{k+1} = \int_0^1 (k+1)(1-t)^k \sum_{|\alpha| = k+1} \partial^\alpha f(x_0 + th) \frac{h^\alpha}{\alpha!} dt$$
$$= O(h^{k+1}) \text{ as } h \to 0.$$

Proof. Since U is open $\exists \varepsilon > 0$ such that $x + th \in U$ for all $t \in (-\varepsilon, 1 + \varepsilon)$. By Taylor's theorem in one variable, applied to $\varphi(t) : (-\varepsilon, 1+\varepsilon) \to \mathbb{R}, t \mapsto f(x_0 + th)$, we have the Taylor approximation for $\varphi(1)$ around 0 with remainder term given by

$$\varphi(1) = \sum_{m=0}^k \frac{\varphi^{(m)}(0)}{m!} 1^m + \int_0^1 \varphi^{(k+1)}(t) \frac{(1-t)^k}{k!} dt.$$

Applying the chain rule to φ, we get for $t \in (-\varepsilon, 1 + \varepsilon)$,

$$\varphi'(t) = \sum_{i=1}^n \partial_i f(x_0 + th) h_i = \sum_{|\alpha| = 1} \partial^\alpha f(x_0 + th) h^\alpha.$$

We now want to show that

$$\varphi^{(m)}(t) = m! \sum_{|\alpha| = m} \partial^\alpha f(x_0 + th) \frac{h^\alpha}{\alpha!}.$$

Indeed, by chain rule and induction over m, we have

$$\varphi^{(m+1)}(t) = \frac{\mathrm{d}}{\mathrm{d}t} \left(m! \sum_{|\alpha| = m} \partial^\alpha f(x_0 + th) \frac{h^\alpha}{\alpha!} \right)$$
$$= m! \left(\sum_{|\alpha| = m} \sum_{i=1}^n \partial_i \partial^\alpha f(x_0 + th) h_i \frac{h^\alpha}{\alpha!} \right)$$

.

Now, if $|\alpha| = m$, then $\partial_i \partial^\alpha f = \partial^{\alpha + e_i} f$, where e_i is the multi-index with 1 in the i-th position and 0 elsewhere. Thus, we can write

$$\varphi^{(m+1)}(t) = m! \left(\sum_{|\beta| = m+1} \partial^\beta f(x_0 + th) h^\beta \sum_{1 \leq i \leq n, \beta_i \geq 1} \frac{\beta_i}{\beta!} \right).$$

Where we used that

$$\frac{1}{\alpha!} = \frac{\alpha_i + 1}{\alpha_1! \cdots (\alpha_i + 1)! \cdots \alpha_n!} = \frac{\beta_i}{\beta!}.$$

Putting things together, we get our desired result. □

Corollary 2.27:

Let $x_0 \in U$, $f \in C^{k+1}(U)$ and $P(x)$ be a polynomial of degree $k \geq 0$. Assume that $|f(x_0 + h) - P(h)| = o(|h|^k)$ as $h \to 0$. Then, P is the Taylor polynomial of f at x_0 of degree k.

Mathematically: $\forall |\alpha| \leq k, \partial^\alpha f(x_0) = \partial^\alpha P(0)$.

Proof. By Taylors theorem, we get

$$\left| P(h) - \sum_{|\alpha| \leq k} \partial^\alpha f(x_0) \frac{h^\alpha}{\alpha!} + R_{k+1} f(x_0, h) \right| = o(|h|^k).$$

But two polynomials of degree k which differ by $o(|h|^k)$ must be equal. (Exercise) □

Example 2.28:

We want to compute the Taylor expansion of degree 2 of

$$f(x, y) = \sqrt{1 + x - y^2}.$$

From analysis I, we know that

$$\sqrt{1+t} = 1 + \frac{t}{2} - \frac{t^2}{8} + O(t^3) \text{ as } t \to 0.$$

Plugging in $t = x - y^2$, we get

$$f(x,y) = 1 + \frac{x - y^2}{2} - \frac{(x - y^2)^2}{8} + O((x - y^2)^3).$$

Expanding this and neglecting terms of degree higher than 2, we get

$$f(x,y) = 1 + \frac{x}{2} - \frac{y^2}{2} - \frac{x^2}{8} + O(r^3) \text{ as } r \to 0.$$

For Taylor approximations to be useful, we need to be able to control the remainder term. This is described by analytic functions.

Definition 2.29: Real analytic functions

Given $U \subset \mathbb{R}^n$ open, we say that $f \in C^\infty(U)$ satisfies analytic estimates in U if for every $x_0 \in U$, there exists $\rho > 0$ and $C > 0$ such that

$$\max_{B_\rho(x_0)} |\partial^\alpha f| \le C|\alpha|!(n\rho)^{-|\alpha|} \text{ for all } \alpha \in \mathbb{N}^n.$$

Thanks to Taylors theorem, a function is analytic iff it can be written as a power series around any point in its domain.

Theorem 2.30: Analytic Expansion

Given $U \subset \mathbb{R}^n$ open, $f : U \to \mathbb{R}$, analytic.

Given $x_0 \in U$, let ρ and C be as in definition 2.29. Then, let

$$P_{x_0,k}(h) \coloneqq \sum_{|\alpha|=1}^{k} \partial^\alpha f(x_0)\frac{h^\alpha}{\alpha!}.$$

Then for all $r \in (0, \rho)$, the power series converges absolutely, i.e. $\forall\ k < l$

$$\sup_{h\in B_r(0)} \sum_{|\alpha|=k}^{l} \left|\partial^\alpha f(x_0)\frac{h^\alpha}{\alpha!}\right| \le \frac{C(\frac{r}{\rho})^k}{1-(\frac{r}{\rho})}.$$

The series converges for $h \in \overline{B_r(0)}$ and we have

$$\sup_{h\in B_r(0)} |P_{x_0,k}(h) - P_{x_0}(h)| \le \frac{C(\frac{r}{\rho})^k}{1-(\frac{r}{\rho})}.$$

Moreover,

$$f(x_0 + h) = P_{x_0,\infty}(h) \quad \forall h \in B_\rho(0).$$

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Proof. The key idea is that for $x \in B_r(x_0), 0 < r < \rho$, we have

$$\begin{aligned}\sum_{|\alpha|=m} \left|\frac{\partial^\alpha f(x)}{\alpha!}h^\alpha\right| &\le \sum_{|\alpha|=m} \frac{|\partial^\alpha f(x)|}{\alpha!}r^m \\ &\le \sum_{|\alpha|=m} C\frac{\rho^{-m}n^{-m}r^m}{\alpha!}|\alpha|! \\ &= C\left(\frac{r}{\rho}\right)^m n^{-m}\sum_{|\alpha|=m}\frac{|\alpha|!}{\alpha!} \quad = C\left(\frac{r}{\rho}\right)^m.\end{aligned}$$

The last step follows since

$$(1 + \cdots + 1)^m = \sum_{|\alpha|=m}\frac{|\alpha|!}{\alpha!}.$$

Thus for the tail of the series,

$$\sum_{|\alpha|=k}^{\infty} \left|\frac{\partial^\alpha f(x_0)}{\alpha!}h^\alpha\right| \le \sum_{m=k}^{\infty} C\left(\frac{r}{\rho}\right)^m = \frac{C(\frac{r}{\rho})^k}{1-(\frac{r}{\rho})}.$$

So $P_{x_0,\infty}(h) = \lim_{k\to\infty} P_{x_0,k}(h)$ exists for $h \in B_\rho(x_0)$.

The error is given by Taylor, $f(x_0 + h) = P_{x_0,k}(h) + R^f_{x_0,k+1}$.

$$\begin{aligned}|R^f_{x_0,k+1}(h)| &= \left|\int_0^1 (k+1)(1-t)^k \sum_{|\alpha|=k+1} \partial^\alpha f(x_0+th)\frac{h^\alpha}{\alpha!}dt\right| \\ &\le \int_0^1 (k+1)(1-t)^k \sum_{|\alpha|=k+1} \left|\partial^\alpha f(x_0+th)\frac{h^\alpha}{\alpha!}\right|dt \\ &\le \int_0^1 (k+1)(1-t)^k C\left(\frac{r}{\rho}\right)^{k+1} dt \le C\left(\frac{r}{\rho}\right)^{k+1}.\end{aligned}$$

As a consequence, by triangle inequality,

$$\begin{aligned}&|f(x_0+h) - P_{x_0,\infty}(h)| \\ &\le \underbrace{|f(x_0+h) - P_{x_0,k}(h)|}_{\text{Rest}} + \underbrace{|P_{x_0,k}(h) - P_{x_0,\infty}(h)|}_{\text{Tail}} \\ &\le C\left(\frac{r}{\rho}\right)^{k+1} + \frac{C(\frac{r}{\rho})^k}{1-(\frac{r}{\rho})} \to 0 \text{ as } k \to \infty.\end{aligned}$$

□

Corollary 2.31: Unique Continuation Principle

Given $U \subset \mathbb{R}^n$ open and connected, f, g analytic in U such that $\exists x_0 \in U$ with

$$\partial^\alpha f(x_0) = \partial^\alpha g(x_0) \quad \forall \alpha \in \mathbb{N}^n.$$

Then $f = g$ on U.

In particular, if $V \subset U$ is an open subset such that $f = g$ on V, then $f = g$ on U.

Proof. Let $x_0 \in G \coloneqq \{x \in U \mid \partial^\alpha f(x) = \partial^\alpha g(x) \forall \alpha \in \mathbb{N}^n\}$. We will show that $U \setminus G$ and G are both open in U. Since U is connected and $x_0 \in G$, we must have $G = U$.

i. $U \setminus G$ is open in U: By definition of G,

$$\begin{aligned}G &= \bigcap_{\alpha\in\mathbb{N}^n} \{x \in U \mid \partial^\alpha(f-g)(x) = 0\} \\ U \setminus G &\overset{\text{De Morgan}}{=} \bigcup_{\alpha\in\mathbb{N}^n} \{x \in U \mid \partial^\alpha(f-g)(x) \ne 0\} \\ &= \bigcup_{\alpha\in\mathbb{N}^n} [\partial^\alpha(f-g)]^{-1}(\mathbb{R}\setminus\{0\}).\end{aligned}$$

Since $\mathbb{R}\setminus\{0\}$ is open and $\partial^\alpha(f-g)$ is continuous, $[\partial^\alpha(f-g)]^{-1}(\mathbb{R}\setminus\{0\})$, the preimage is open. Since arbitrary unions of open sets are open, $U\setminus G$ is open.

ii. G is open in U: Let $x \in G \subset U$. Since f, g are analytic, $\exists \rho > 0$ such that $\forall h \in B_\rho(x)$,

$$f(x+h) = P^f_{x,\infty}(h), g(x+h) = P^g_{x,\infty}(h).$$

But $P^f = P^g$ since Taylor polynomials are defined by the derivatives at $x \in G$.

Hence, $f \equiv g$ on $B_\rho(x)$, so $B_\rho(x) \subset G$, so G is open. □

3 Optimization

In this chapter we want to treat the standard problem of minimizing or maximizing a function $f(x_1, \dots, x_n) \in \overline{U}$ with constraints $g_k(x_1, \dots, x_n) = 0$ for $k = 1, \dots, m$.

3.1 First and Second Order Optimality Conditions

To describe the optimality conditions, we need to introduce some definitions.

Definition 3.1: Local Minimum / Maximum

Given $U \subset \mathbb{R}^n$ open, $f \in C^1(U)$. We say that x_0 is a LOCAL MINIMUM of f if $\exists r > 0$ such that

$$f(x_0) \leq f(x) \text{ for all } x \in B_r(x_0).$$

LOCAL MAXIMUM of f if $\exists r > 0$ such that

$$f(x_0) \geq f(x) \text{ for all } x \in B_r(x_0).$$

STRICT LOCAL MINIMUM of f if $\exists r > 0$ such that

$$f(x_0) < f(x) \text{ for all } x \in B_r(x_0) \setminus \{x_0\}.$$

If x_0 is a local minimum or maximum of f, then x_0 is called a LOCAL EXTREMUM of f.

Definition 3.2: Critical Point

Given $U \subset \mathbb{R}^n$ open, $f \in C^1(U)$. Then $x_0 \in U$ is a CRITICAL POINT of f if $\nabla f(x_0) = 0$.

In other words, $\partial_i f(x_0) = 0$ for all $i = 1, \dots, n$.

Proposition 3.3:

Every local Extremum is a critical point.

Proof. Assume wlog x_0 is a local minimum. Fix $i \in 1, \dots, n$. Then

$$\partial_i f(x_0) = \lim_{h \to 0} \frac{f(x_0 + te_i) - f(x_0)}{t}.$$

Letting t approach 0 from both sides, we get

$$\lim_{t \to 0+} \frac{f(x_0 + te_i) - f(x_0)}{t} \geq 0$$
$$\lim_{t \to 0-} \frac{f(x_0 + te_i) - f(x_0)}{t} \leq 0.$$

Since the limit exists, we must have $\partial_i f(x_0) = 0$. □

Theorem 3.4: Lagrange Multipliers

Given $U \subset \mathbb{R}^n$ open, $f, g_j \in C^1(U)$ for $j = 1, \dots, m$. Let $M = \{x \in U \mid g_1(x) = \dots = g_m(x) = 0\}$ be the constraint set. If x_0 is a local minimum of $f|_M$, then $\exists \lambda_*, \lambda_1, \dots, \lambda_m \in \mathbb{R}$, with $\lambda_*^2 + \lambda_1^2 + \dots + \lambda_m^2 = 1$, such that

$$\lambda_* \nabla f(x_0) + \sum_{j=1}^m \lambda_j \nabla g_j(x_0) = 0.$$

$\lambda_*, \lambda_1, \dots, \lambda_m$ are called the LAGRANGE MULTIPLIERS.

Proof. First assume that x_0 is a strict local minimum.

For $\varepsilon > 0$, let $f_\varepsilon(x) = f(x) + \frac{1}{2\varepsilon} \sum_{j=1}^m g_j(x)^2$, defined for $x \in \overline{B_{\frac{r}{2}}(x_0)}$, where r is such that $f(x) > f(x_0)$ for all $x \in B_r(x_0) \setminus \{x_0\}$.[4] Let $\varepsilon_l = \frac{1}{l}$. Choose x_l as a point of minimum of the continuous function f_{ε_l} on the compact set $\overline{B_{\frac{r}{2}}(x_0)}$.

We claim that some subsequence x_{l_k} converges to x_0. Indeed, by compactness, we can choose a subsequence x_{l_k} converging to some $\overline{x} \in \overline{B_{\frac{r}{2}}(x_0)}$. Since x_l is a minimum we have

$$f_{\varepsilon_l}(x_l) \leq f_{\varepsilon_l}(x_0) = f(x_0).$$

So also

$$\frac{1}{2\varepsilon_l} \sum_{j=1}^m g_j(x_l)^2 \leq f_{\varepsilon_l}(x_l) \leq f(x_0).$$

This implies that $\sum_{j=1}^m g_j(x_l)^2 \leq 2\varepsilon_l f(x_0) \to 0$ as $l \to \infty$, so $\sum_{j=1}^m g_j(\overline{x})^2 = 0$, so $\overline{x} \in M$. From this, we get

$$f(x_{l_k}) \leq f_{\varepsilon_{l_k}}(x_{l_k}) \leq f(x_0).$$

Since $x_{l_m} \to \overline{x}$, by continuity of f we get $f(\overline{x}) \leq f(x_0)$. Since x_0 is a strict local minimum, we must have $\overline{x} = x_0$. Lec 13

Let $y_m = x_{l_m}$ and $\tilde{\varepsilon}_m = \varepsilon_{l_m}$. Since $y_m \in B_r(x_0)$ (for large m, y_m cannot be on the boundary), is an interior minimum of $f_{\tilde{\varepsilon}_m}$, we have

$$0 = \tilde{\varepsilon}_m \nabla f_{\tilde{\varepsilon}_m}(y_m) = \tilde{\varepsilon}_m \nabla f(y_m) + \sum_{j=1}^m g_j(y_m) \nabla g_j(y_m).$$

Let $\mu_m^2 = \tilde{\varepsilon}_m^2 + \dots + [g_m(y_m)]^2 > 0$. Then,

$$\left(\frac{\tilde{\varepsilon}_m}{\mu_m}\right)^2 + \dots + \left(\frac{g_m(y_m)}{\mu_m}\right)^2 = 1.$$

Define

$$\lambda_0^m = \frac{\tilde{\varepsilon}_m}{\mu_m}, \lambda_j^m = \frac{g_j(y_m)}{\mu_m} \text{ for } j = 1, \dots, m.$$

So there exists $\lambda^m = (\lambda_0^m, \lambda_1^m, \dots, \lambda_j^m) \in S^j$ i.e.

$$(\lambda_0^m)^2 + \dots + (\lambda_m^m)^2 = 1.$$

Hence,

$$0 = \lambda_0^m \nabla f(y_m) + \sum_{j=1}^m \lambda_j^m \nabla g_j(y_m).$$

Since S^j is closed and bounded, it is compact, Hence, there exists a subsequence λ^{m_k} converging to some $\lambda_* = (\lambda_*, \lambda_1, \dots, \lambda_m) \in S^j$. Since $y_{m_k} \to x_0$, by continuity of ∇f and ∇g_j, we get

$$0 = \lambda_* \nabla f(x_0) + \sum_{j=1}^m \lambda_j \nabla g_j(x_0).$$

□

Theorem 3.5: Spectral Theorem

If A is $n \times n$ symmetric ($A^T = A$), with coefficients in $\mathbb{R}$, then A diagonalizes in some orthonormal basis and has real eigenvalues.

Also, for $O = (v_1 | \dots | v_n)$, where $v_1, \dots, v_n$ is an orthonormal basis of eigenvectors of A, we have

$$O^T O = I_n, O^{-1} = O^T, O^{-1} A O = \Lambda = \begin{pmatrix} \lambda_1 & \cdots \\ \vdots & \ddots \end{pmatrix}.$$

[4] This function is called penalizing function as for points not satisfying the constrains, it diverges to $+\infty$ as $\varepsilon \to 0$.

Lemma 3.6:

Assume $v_1, \dots, v_k$, with $0 \le k < n$, are vectors such that $v_i \cdot v_j = \delta_{ij}$. Let $A \in M_{n\times n}(\mathbb{R})$ be symmetric.

If there exist $\mu_i \in \mathbb{R}$ such that $Av_i = \mu_i v_i$ then there exists $w \in \mathbb{R}^n$ such that $|w| = 1$, $w \cdot v_i = 0$ and $Aw = \lambda w$ for some $\lambda \in \mathbb{R}$.

Proof. Let $f : \mathbb{R}^n \to \mathbb{R}$, $x \mapsto x \cdot Ax$. We want to minimize this function under the constraints

$$\begin{aligned} g_*(x) &= |x|^2 - 1 = 0 \\ g_1(x) &= v_1 \cdot x = 0 \\ &\vdots \\ g_k(x) &= v_k \cdot x = 0. \end{aligned}$$

We apply theorem 3.4 with

$$M = x \in \mathbb{R}^n \mid g_*(x) = \dots = g_k(x) = 0\} \subset S^{n-1}.$$

Note that $M \subset \mathbb{R}^n$ is closed and bounded, hence compact.

Since f is continuous, there exists $w \in M$ as a point of minimum of $f|_M$. Hence, $\exists \lambda_0, \lambda_*, \lambda_1, \dots, \lambda_k \in \mathbb{R}$, with $\sum_i \lambda_i^2 = 1$, such that

$$\lambda_0 \nabla f(w) + \lambda_* \nabla g_*(w) + \dots + \lambda_k \nabla g_k(w) = 0.$$

Note that in general, for symmetric matrices

$$v \cdot w = v^T w \Rightarrow v \cdot Aw = Aw \cdot v = (Av) \cdot w = w \cdot Av.$$

Hence, calculating the partial derivatives, we get

$$\begin{aligned} \partial_i f(x) &= \lim_{t\to 0} \frac{f(x+te_i)}{t} \\ &= \lim_{t\to 0} \frac{(x+te_i)\cdot A(x+te_i) - x\cdot Ax}{t} \\ &= \lim_{t\to 0} \frac{2te_i \cdot Ax + t^2 e_i \cdot Ae_i}{t} = 2e_i \cdot Ax \\ &\Rightarrow \nabla f(x) = 2Ax. \end{aligned}$$

Replacing A with I, the above calculation gives $\nabla g_*(x) = 2x$. Also

$$\partial_i g_j(x) = \lim_{t\to 0} \frac{v_j \cdot (x+te_i) - v_i x}{t} = v_j \cdot e_i \Rightarrow \nabla g_j(x) = v_j.$$

Plugging this into the Lagrange multipliers condition, we get

$$\underbrace{2\lambda_0 Aw + 2\lambda_* w}_{w_1} + \underbrace{\lambda_1 v_1 + \dots + \lambda_k v_k}_{w_2} = 0.$$

Since $w \in M$, we have $w \cdot v_j = 0 \forall j = 1, \dots, k$. This implies that

$$v_j \cdot Aw = w \cdot Av_j = w \cdot \mu_j v_j = \mu_j w \cdot v_j = 0.$$

This means that Aw is perpendicular to $v_1, \dots, v_k$. Since w_2 is a linear combination of $v_1, \dots, v_k$, we get $w_1 \cdot w_2 = 0$. Hence, $w_1 + w_2 = 0$ implies $w_1 = w_2 = 0$. If the sum of two perpendicular vectors is 0, then both vectors must be 0. Hence $2\lambda_0 Aw + 2\lambda_* w = 0$, and $\lambda_i = 0$ since $w_2 = 0$ and w_2 is a linear combination of $v_1, \dots, v_k$ which are linearly independent.

By definition of the Lagrange multipliers, $\lambda_0^2 + \lambda_*^2 = 1$. Hence, $Aw = \lambda w$ for $\lambda = -\frac{\lambda_*}{\lambda_0} \in \mathbb{R}$. Note that $\lambda_0 \neq 0$ since otherwise $\lambda_*^2 = 1$, so $\lambda_* = \pm 1$ and $w = 0$, contradicting $|w| = 1$. □

Proof. [Theorem 3.5] By induction over k, we can construct an orthonormal basis of eigenvectors of A. □

For second order optimality conditions, we need to introduce the following definition.

Definition 3.7: Hessian

Given $U \subset \mathbb{R}^n$ open, $f : U \to \mathbb{R} \in C^2$. The Hessian Matrix of f at $x \in U$ is the $n \times n$ matrix defined by

$$H_{ij} f(x) = \partial_i \partial_j f(x).$$

By Schwarz's theorem, the Hessian is a symmetric matrix. It is also denoted $D^2 f(x)$. The Laplacian of f is the trace of the Hessian:

$$\Delta f(x) = \operatorname{tr}(Hf(x)) = \sum_{i=1}^n \partial_{ii} f(x).$$

Definition 3.8:

$A \in M_{n\times n}(\mathbb{R})$ is non-negative definite if $\forall x \in \mathbb{R}^n, x \cdot Ax \ge 0$.

Proposition 3.9:

Given $U \subset \mathbb{R}^n$ open, $f \in C^3(U)$. If f has a local minimum at $x_0 \in U$, then $Hf(x_0)$ is non-negative definite.

Proof. By Taylor, $f(x_0 + x) = f(x_0) + \nabla f(x_0)\cdot x + \frac{1}{2} x \cdot Hf(x_0) x + O(|x|^3)$ as $x \to 0$.

By the spectral theorem, $O^T H O = \Lambda$.

Let $y = Ox \in \mathbb{R}^n$. Then

$$f(x_0 + x) = f(x_0) + \underbrace{\nabla f(x_0) \cdot x}_{=0} + \frac{1}{2} y \cdot \Lambda y + O(|y|^3) \text{ as } y \to 0.$$

Since x_0 is a local minimum, we must have $y \cdot \Lambda y \ge 0$ for all $y \in \mathbb{R}^n$. This implies that Λ is non-negative definite, so $Hf(x_0)$ is non-negative definite. □ Lec 14

Definition 3.10:

A matrix A is called

- positive definite if $\min\{\lambda_i\} > 0$
- nonnegative definite if $\min\{\lambda_i\} \ge 0$
- negative definite if $\max\{\lambda_i\} < 0$
- non-positive definite if $\max\{\lambda_i\} \le 0$
- indefinite if $\min\{\lambda_i\} < 0 < \max\{\lambda_i\}$

Proposition 3.11: Hessian test

Let $x_0 \in U$ be a critical point of $f \in C^3(U)$.

1) x_0 local minimum $\Rightarrow Hf(x_0)$ non-negative definite.

2) $Hf(x_0)$ positive definite $\Rightarrow x_0$ local minimum.

3) x_0 local maximum $\Rightarrow Hf(x_0)$ non-positive definite.

4) $Hf(x_0)$ negative definite $\Rightarrow x_0$ local maximum.

5) $Hf(x_0)$ indefinite $\Rightarrow x_0$ is neither local minimum nor local maximum.

Proof. We will only show 2) and 3).

2) By Taylor, $f(x_0+t\underbrace{Oy}_{=x}) = f(x_0)+\frac{1}{2}\sum_{i=1}^n \lambda_i y_i^2 + O(|y|^3)$ as $y \to 0$. Let $\lambda = \min\{\lambda_i\} > 0$. Then, for $|y| \le r_0$ we have

$$\begin{aligned} f(x_0+tOy) &\ge f(x_0) + \frac{1}{2}\sum_{i=1}^n \lambda y_i^2 - C|y|^3 \\ &\ge f(x_0) + \frac{1}{2}\lambda |y|^2 - C|y|^3 \end{aligned}$$

.

We can choose r small enough such that $\frac{1}{2}\lambda r^2 - Cr^3 > 0$. Then $\forall y \in B_r(x_0)$, $f(x_0+x) > f(x_0)$, so x_0 is a local minimum.

3) Assume by contradiction $\exists \lambda_j > 0$. Then by Taylor,

$f(x_0+tOe_i) \ge f(x_0) + \lambda_j t^2 + O(t^3) > f(x_0)$ for $t > 0$ small enough.

This contradicts the fact that x_0 is a local maximum. □

Theorem 3.12: Fundamental Theorem of Algebra

A polynomial of degree n with complex coefficients has n complex roots, up to repetitions.

Proof. We show that there exists one root. Dividing and inductively applying the fundamental theorem of algebra to the quotient, we get the desired result.

Let $P(t) = a_0 + a_1 t + \dots + a_n t^n$ be a polynomial of degree n with $a_i \in \mathbb{C}$ and $a_n = 1$. We want to find t_0 such that $|P(t_0)| = 0$. We thus try to minimize $|P(z)|$ for $z \in \mathbb{C}$. The issue here is, that $\mathbb{C}$ is not compact, so we cannot apply Weierstrass theorem.

Step 1: Let $\beta = \inf_{z\in\mathbb{C}} |P(z)|$. We show that $|P(z)| \to \infty$ as $|z| \to \infty$. Indeed,

$$\begin{aligned} P(z) &= a_0 + a_1 z + \dots + a_{n-1}z^{n-1} + z^n \\ &\overset{\Delta}{\ge} |z|^n - (|a_0| + |a_1||z| + \dots + |a_{n-1}||z|^{n-1}) \\ &\ge R^n - CR^{n-1} \text{ for } |z| \ge R \\ &\ge 1 + 10\beta > \beta \text{ for } R \text{ large enough.} \end{aligned}$$

Here we defined $C = |a_0| + |a_1| + \dots + |a_{n-1}|$.

Hence, the Infimum is attained in $\overline{B_R(0)}$ so $\exists z_0 \in \overline{B_R(0)}$ such that $|P(z_0)| = \beta$. This implies that

$$\inf_{\mathbb{C}} |P(z)| = \min_{\overline{B_R(0)}} |P(z)|,$$

since P is continuous and $\overline{B_R(0)}$ is compact.

Since we had a strict inequality, $z_0 \in B_R(0)$, i.e. z_0 does not lie on the boundary. If $\beta = 0$, then $P(z_0) = 0$ so we have a root.

Assume thus, $\beta > 0$. Then $P(z_0+z)$ has a minimum at $z = 0$. By Taylor, we can write

$$P(z_0+z) = \sum_{k=1}^n a_k (z_0+z)^k = \underbrace{\sum_{k=1}^n b_k z^k}_{:=Q}, b_k \in \mathbb{C}.$$

Notice, that $|b_0| = |Q(0)| = |P(z_0)| = \beta > 0$. Let $l = \min\{k \ge 1 \mid b_k \ne 0\}$, i.e. l is the order of the first non-zero term in the Taylor expansion of $P(z_0+z)$ around $z = 0$. Then

$$\begin{aligned} |P(z_0+z)| &= |b_0 + b_l z^l + \dots| \\ &= |b_0| \left|1 + \frac{b_l}{b_0} z^l + \dots\right| \\ &= |b_0| \left|c_l z^l + \sum_{k=1+l}^n c_k z^k\right| \text{ for } c_k = \frac{b_k}{b_0}. \end{aligned}$$

We have a contradiction if $c_l z^l < 0$. Note that we must have

$$c_l z^l = \rho e^{i\alpha} |z|^l e^{il\arg(z)} \in \mathbb{R}.$$

This is exactly the case if $\alpha + l\arg(z) = \pi$. Thus, we choose γ accordingly (i.e. $\alpha + l\gamma = \pi$) and let $z = re^{i\gamma}$ for $r > 0$ small enough. Then

$$\begin{aligned} |P(z_0+z)| &= |b_0| \left|1 + \rho r^l e^{i\pi} + O(r^{l+1})\right| \\ &\le |b_0| \left|1 - \rho r^l + Cr^{l+1}\right| \\ &< |b_0| = \beta \text{ for } r \text{ small enough.} \end{aligned}$$

This contradicts the fact that $z = 0$ is a minimum of $P(z_0+z)$, so we must have $\beta = 0$, so $P(z_0) = 0$ and z_0 is a root of P. □

Definition 3.13: Convex Functions

Let $A \subset \mathbb{R}^n$ be a convex set. $f : A \to \mathbb{R}$ is CONVEX if $\forall x, y \in A, t \in [0,1]$,

$$f((1-t)x + ty) \le (1-t)f(x) + tf(y).$$

Proposition 3.14:

Let $f \in C^2(U)$, where $U \subset \mathbb{R}^n$ is open and convex. Then f is convex iff $Hf(x)$ is non-negative definite for all $x \in U$ iff $\forall x, y \in U$,

$$f(y) - f(x) \ge Df_x(y-x).$$

Theorem 3.15: Jensen's Inequality

Let $w_1, \dots, w_N \in [0,1]$ with $\sum_{i=1}^N w_i = 1$ and $x_1, \dots, x_N \in U$. $f : U \to \mathbb{R}$ is convex iff

$$f\left(\sum_{i=1}^N w_i x_i\right) \le \sum_{i=1}^N w_i f(x_i).$$

Proof. [of Proposition 3.14] Given $x, y \in U$, let

$g(s) = f((1-s)x + sy) = f(x_0 + vs)$ for $x_0 = x, v = y - x$.

$1 \Rightarrow 2)$ g is convex if

$g((1-t)a + tb) \le (1-t)g(a) + tg(b)$ for $a, b \in [0,1], t \in [0,1]$.

Since $g \in C^2$, from analysis I we know that g is convex iff $g''(s) \ge 0$ for all $s \in [0,1]$. But then $0 \le g''(s) = \frac{1}{2} v \cdot Hf(x_0)v$. So, for all $v \in \mathbb{R}^n$, $v \cdot Hf(x_0)v \ge 0$, which is equivalent to $Hf(x_0)$ being non-negative definite.

$2 \Rightarrow 3)$ We know by Taylor that

$$g(1) = g(0) + g'(0) + \int_0^1 (1-s)g''(s)ds.$$

Since $g''(s) \ge 0$, we have $\int_0^1 (1-s)g''(s)ds \ge 0$. Hence,

$$f(x_0+v) \ge f(x_0) + Df_{x_0}(v).$$

Let $y = x_0 + v, x = x_0$. Then we get the desired result.

$3 \Rightarrow 1)$ We show that the last property implies the Jensen inequality. We get that

$$f(x) \ge f(x_0) + Df_{x_0}(x - x_0).$$

So by linearity of Df_{x_0},

$$\begin{aligned} w_i f(x) &\ge w_i f(x_0) + Df_{x_0}(w_i(x-x_0)) \text{ for } i = 1, \dots, N \\ \Rightarrow \sum_{i=1}^N w_i f(x) &\ge \sum_{i=1}^N w_i f(x_0) + \sum_{i=1}^N Df_{x_0}(w_i(x-x_0)) \\ &= \sum_{i=1}^N w_i f(x_0) + Df_{x_0}\Bigg(\underbrace{\sum_{i=1}^N w_i(x-x_0)}_{=0}\Bigg) \end{aligned}$$

$$\sum_{i=1}^N w_i f(x) \ge \sum_{i=1}^N w_i f(x_0) = f(x_0).$$

□

4 Inverse and Implicit Function Theorems and Submanifolds

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4.1 Inverse Function Theorem

We consider the following question: Can a map from $\mathbb{R}^n \to \mathbb{R}^n$ that is a small perturbation of the identity map be inverted?

Lemma 4.1:
Let $U \subset \mathbb{R}^n$ be open, $F : U \to \mathbb{R}^n$ a function of the form

$$F(x) = x + \phi(x),$$

where ϕ is a λ-Lipschitz function for $\lambda \in (0, 1)$. Then

1) $\forall x_0 \in U$, and $r > 0$ such that $B_r(x_0) \subset U$,

$$B_{(1-\lambda)r}(F(x_0)) \subset F(B_r(x_0)) \subset F(U).$$

In particular, $F(U)$ is open.

2) $F : U \to F(U)$ is bijective and F^{-1} is $\frac{1}{1-\lambda}$-Lipschitz.

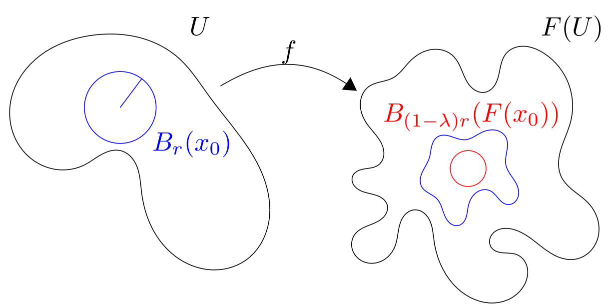

Figure 9: A λ-Lipschitz perturbation of the identity map is invertible.

The idea of the proof is: Given $y \in B_{(1-\lambda)r}(F(x_0))$, we want to find $x \in B_r(x_0)$ such that

$$F(x) =< \Leftrightarrow x + \phi(x) = y \Leftrightarrow x = y - \phi(x) =: T(x).$$

Note that T is a contraction. The problem is that T does not map U to itself. We thus have to modify Banach's fixed point theorem.

Proof. 1) Choose $y \in B_{(1-\lambda)r}(F(x_0))$. We want to find $x \in B_r(x_0)$ such that $F(x) = y$. Choose x_0 and define $x_{k+1} = T(x_k) := y - \phi(x_k)$.

So $x_1 = T(x_1) = y - \phi(x_0)$. This implies that

$$|x_1 - x_0| = |y - \phi(x_0) - x_0| = |y - F(x_0)| < (1 - \lambda)r.$$

It follows, that $x_1 \in U$. Hence, $x_2 = T(x_1)$ is well-defined.

In general, we have

$$|x_{k+1} - x_k| \leq \lambda^k |x_1 - x_0| \leq \lambda^k (1 - \lambda)r.$$

So it follows that

$$|x_{k+1} - x_0| \leq \sum_{j=0}^{k} |x_{j+1} - x_j| \leq (1 - \lambda)r \sum_{j=0}^{\infty} \lambda^j = r.$$

So for all $k \in \mathbb{N}$, $x_k \in U$. Analogously to the proof of Banach's fixed point theorem, we can show that x_k is Cauchy and thus converges to x such that $F(x) = y$. Thus,

$$|xk + 1 - x_0| \leq \sum_{i=1}^{k} |x_{i+1} - x_i| = |x_1 - x_0| \sum_{i=0}^{k-1} \lambda^i \leq r.$$

Hence, it follows that

$$|x - x_0| < r \Rightarrow x \in B_r(x_0).$$

2) We show that F is injective. Let $x, x' \in U$ such that $F(x) = F(x')$. Then

$$x + \phi(x) = x' + \phi(x') \Leftrightarrow |x - x'| = |\phi(x) - \phi(x')| \leq \lambda |x - x'|.$$

Since $\lambda < 1$, we must have $x = x'$. Hence, F is injective.

Since by definition, F is surjective, it is bijective.

We now show that F^{-1} is $\frac{1}{1-\lambda}$-Lipschitz. Let $F(x) = y, F(x') = y'$. Then

$$y - y' = x - x' + \phi(x) - \phi(x').$$

This implies that

$$\begin{aligned} |y - y'| &\geq |x - x'| - |\phi(x) - \phi(x')| \\ &\geq |x - x'| - \lambda |x - x'| \\ &= (1 - \lambda)|x - x'|. \end{aligned}$$

Plugging in $x = F^{-1}(y), x' = F^{-1}(y')$, we get

$$|F^{-1}(y) - F^{-1}(y')| \leq \frac{1}{1 - \lambda} |y - y'|.$$

□

Lemma 4.2: Automatic differentiablity of the inverse
Let $U, V \subset \mathbb{R}^n$ be open, $f : U \to V, g : V \to U$ be bijective, such that $f(g(y)) = y \, \forall y \in V$. Assume, that f is differentiable at $x_0 \in U$ and $Df_{x_0} : \mathbb{R}^n \to \mathbb{R}^n$ is invertible and g is Lipschitz.

Then, g is differentiable at $y_0 = f(x_0)$ and $Dg_{y_0} = (Df_{x_0})^{-1}$.

Proof. By definition, $f(x) - f(x_0) = Df_{x_0}(x - x_0) + o(|x - x_0|)$ as $x \to x_0$. Thus,

$$f(g(y)) - f(g(y_0)) = \underbrace{Df_{x_0}}_{:=L}(g(y) - g(y_0)) + o(|g(y) - g(y_0)|) \text{ as } y \to y_0.$$

Multiplying both sides by L^{-1}, we get

$$g(y) - g(y_0) = L^{-1}(y - y_0) + o(L^{-1}|g(y) - g(y_0)|) \text{ as } y \to y_0.$$

If Λ is the Lipschitz constant of g, then $|g(y) - g(y_0)| \leq \Lambda |y - y_0|$. Hence,

$$\begin{aligned} g(y) - g(y_0) &= L^{-1}(y - y_0) + o(\|L^{-1}\|_2 \Lambda |y - y_0|) \\ &= L^{-1}(y - y_0) + o(|y - y_0|). \end{aligned}$$

□

Lemma 4.3:
Let $U \subset \mathbb{R}^{n \times n}$ be the set of invertible functions (which is open by continuity of the determinant). Then,

$$\Theta : U \to U, x \mapsto x^{-1} \text{ is } C^\infty.$$

Proof. Exercise □ Lec 16

Theorem 4.4: Inverse Function Theorem

Let $U \subset \mathbb{R}^n$ be open, $f : U \to \mathbb{R}^n \in C^1(U)$. If Df_{x_0} is invertible for some $x_0 \in U$, then there exists $U_0 \subset U$ open, such that

1. $f|_{U_0}$ is injective,
2. $f(U_0) := V$ is open,
3. $g = f|_{U_0}^{-1} : V \to U_0$ is C^1 and $Dg_{f(x)} = (Df_x)^{-1}$ for all $x \in U_0$.

Moreover, if $f \in C^k(U)$ for some $k \geq 1$, then $g \in C^k(V)$.

Proof. We consider the special case $x_0 = 0, f(x_0) = 0$. The general case is obtained by $\tilde{f}(x) = f(x + x_0) - f(x_0)$.

Heuristically we will show that

$$f(x) \approx \underbrace{Df_0(x)}_{:=L(x)} + o(|x|) \Rightarrow L^{-1} \circ f(x) \approx \mathrm{id}(x) + o(\|L^{-1}\|_2|x|).$$

Precisely, let $L = Df_0, F = L^{-1} \circ f$. By the chain rule, we have

$$DF_0 = DL^{-1}_{f(0)} \circ Df_0 = L^{-1} \circ L = \mathrm{id}\,.$$

Let $\varphi = F - \mathrm{id}$. Then $D\varphi_0 = 0$. Hence, $\|J\varphi(0)\|_2 = 0$. Thus, by continuity of the derivative, $\exists r > 0$ such that $\forall x \in B_r(0)$:

$$\|J\varphi(x)\|_2 \leq \frac{1}{2}.$$

Since bounded derivative on connected sets implies Lipschitz, φ is $\frac{1}{2}$-Lipschitz on $B_r(0)$, i.e.

$$|\varphi(x) - \varphi(y)| < \frac{1}{2}|x - y|.$$

Applying the small perturbation lemma (Lemma 4.1) to φ on $U_0 = B_r(0)$, we get that $F|_{U_0}$ is injective, $F(U_0)$ is open and $F^{-1} : F(U_0) \to U_0$ is 2-Lipschitz.

Since $F = L^{-1} \circ f$, we have $f = L \circ F$.

For 2) $V = f(U_0) = L \circ F(U_0)$ is open since $F(U_0)$ is open and L^{-1} is continuous.

For 1) L and F are injective, so $f = L \circ F$ is injective.

For 3) we want to apply lemma 4.2. We thus need to show that Df_x is invertible for all $x \in U_0$ and that g is Lipschitz.

For the first point, we have $f = l \circ (\mathrm{id} + \varphi)$, so by the chain rule,

$$Df_x = L(\mathrm{id} + D\varphi_x).$$

We show the kernel of this map is trivial. Indeed,

$$|(\mathrm{id} + D\varphi_x)y| = |y + D\varphi_x y| \geq |y| - \|D\varphi_x\|_2|y| \geq \frac{1}{2}|y|.$$

Hence, $Df_x(y) = 0$ iff $y = 0$. So, Df_x is injective and thus invertible.

For the second point, we have $g = F^{-1} \circ L^{-1}$. Since L^{-1} is linear, it is Lipschitz. Since F^{-1} is 2-Lipschitz, g is Lipschitz as well.

Finally we want to show that g is $C^1(V)$. We have

$$Jg(y) = Jf(g(y))^{-1}.$$

Since all these functions are continuous, g is C^1.

If $f \in C^k, Jf \in C^{k-1}$, so we get in the same way, knowing $g \in C^1$, that $Jg \in C^1 \Rightarrow g \in C^2$. Iterating this argument, we get $g \in C^k$. $\square$

Definition 4.5: Diffeomorphism

Let $U, V \subset \mathbb{R}^n$ be open. A map $f : U \to V \in C^{-1}$ is called a DIFFEOMORPHISM if f is bijective and $f^{-1} : V \to U$ is C^1.

If f and f^{-1} are C^k, then f is called a C^k-diffeomorphism.

An important consequence is the implicit function theorem

Theorem 4.6: Implicit Function Theorem

Let $0 < d < n, k \geq 1 \in \mathbb{N}$ and let $U \subset \mathbb{R}^n$ be open, $f \in C^1(U, \mathbb{R}^{n-d})$. We write a point in $\mathbb{R}^n \times \mathbb{R}^{n-d}$ as (x, y), where $x \in \mathbb{R}^d, y \in \mathbb{R}^{n-d}$. If there exists $(x_0, y_0) \in U$ such that $f(x_0, y_0) = 0$ and $J_y f_{(x_0,y_0)}$ is invertible, then for sufficiently small $r, s > 0$ there exists $g : B_r(x_0) \subset \mathbb{R}^d \to B_s(y_0) \subset \mathbb{R}^{n-d}$ such that for all $(x, y) \in B_r(x_0) \times B_s(y_0)$, $f(x, y) = 0$ iff $y = g(x)$.

Moreover, $\forall x \in B_r(x_0)$,

$$Jg(x) = -((J_y f)(x, g(x)))^{-1}(J_x f)(x, g(x)).$$

Furthermore, if $f \in C^k(U, \mathbb{R}^{n-d})$ for some $k \geq 1$, then $g \in C^k(B_r(x_0), \mathbb{R}^{n-d})$.

Proof. Consider the function $\Phi \in C^1(U, \mathbb{R}^n)$ given by

$$\Phi(x, y) = (x, f(x, y)).$$

By assumptions, $J\Phi(x_0, y_0)$ has a block structure of the form

$$J\Phi(x_0, y_0) = \begin{pmatrix} I_d & 0 \\ J_x f(x_0, y_0) & J_y f_{(x_0,y_0)} \end{pmatrix}.$$

Since $J_y f_{(x_0,y_0)}$ is invertible, $J\Phi(x_0, y_0)$ is invertible as well. By the inverse function theorem, Φ has a C^1-inverse when restricted to a small cylinder centered at (x_0, y_0),

$$U_0 := B_r(x_0) \times B_s(y_0) \subset U,$$

for $r, s > 0$, which is mapped to the open set $V := \Phi(U_0) \subset \mathbb{R}^n$.

Let $\Psi : V \to U_0$ denote the inverse of $\Phi|_{U_0}$. For given points $(x, y) \in U_0$, put

$$(\xi, \eta) := \Phi(x, y) = (x, f(x, y)) \Leftrightarrow (x, y) = \Psi(\xi, \eta).$$

Then, $\xi = x$ implying that $\Psi(\xi, \eta)$ is of the form

$$\Psi(\xi, \eta) = (\xi, G(\xi, \eta)) \quad \forall(\xi, \eta) \in V,$$

where $G : V \to B_s(y_0)$ is of class C^1 (or C^k).

Thus, since $(x, y) = \Psi(x, \eta) = (x, G(x, \eta))$, we get

$$f(x, y) = 0 \Leftrightarrow \eta = 0 \Leftrightarrow y = G(x, 0).$$

Where the second $\Leftarrow$ follows as Ψ is bijective.

Thus defining $g(x) := G(x, 0)$ we find what we desired.

Lastly, the formula for $Jf(x_0)$ follows from differentiating the identity, $f(x, g(x)) = 0$ using the chain rule. Indeed, for all $i \leq d$ we get

$$0 = \partial_i f(x, g(x)) = \partial_{x_i} f(x, f(x)) + \sum_{l=1}^{n-d} \partial_{y_l} f(x, g(x))\partial_l g(x).$$

Or in matrix form

$$0 = J_x f(x, g(x)) + J_y f(x, g(x)) J_x g(x).$$

$\square$

4.2 Submanifolds

We begin with the definition

Definition 4.7: Submanifold of $\mathbb{R}^n$

Given $0 < d < n$ and $k \geq 1 \in \mathbb{N}$. We say that $M \subset \mathbb{R}^n$ nonempty is a D-DIMENSIONAL SUBMANIFOLD of $\mathbb{R}^n$ of class C^k if, for every point $p_0 \in M, \exists U \subset \mathbb{R}^n$ open containing p_0, $V \subset \mathbb{R}^n$ open containing 0, and a C^k-diffeomorphism $\Psi : U \to V$, such that:

$$\Psi(M \cap U) = \{y \in V | y_{d+1} = \cdots + y_n = 0\}.$$

The map Ψ is called a SUBMANIFOLD CHART.

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In words, Ψ flattens a certain part of the submanifold. This can be visualized as follows:

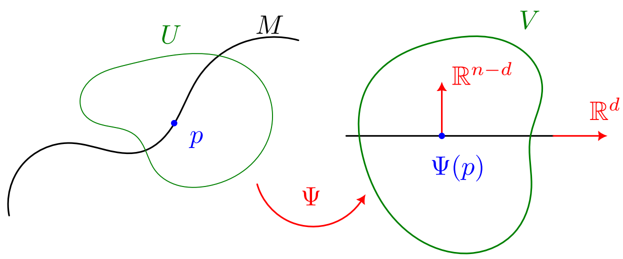

Figure 10: A submanifold chart Ψ flattens a part of the submanifold.

Proposition 4.8: Equivalent Statements for Submanifolds

The following 4 statements are equivalent:

1. (Definition) $M \subset \mathbb{R}^n$ is a d-dimensional submanifold of class C^k.

2. (Implicit) $\forall p \in M$, $\exists U \subset \mathbb{R}^n$ open, such that $p \in U$ and $\exists f : U \to \mathbb{R}^{n-d} \in C^k(U)$ such that Df_p has rank $n - d$ and $M \cap U = \{x \in U | f(x) = 0\}$.

3. (Parametrization) $\forall p \in M$, $\exists V \subset \mathbb{R}^d$ open, $q \in V$, $G : V \to \mathbb{R}^n \in C^k(V)$ such that DG_q has rank d, $G(q) = p$ and $\forall V' \subset V$ open $\exists U' \subset \mathbb{R}^n$ open with $M \cap U' = \{G(y) \mid y \in V\}$.

4. (Graphical) $\forall p \in M$, $\exists U \subset \mathbb{R}^n$ open with $p \in U$, $V \subset \mathbb{R}^d$ open, g: $V \to \mathbb{R}^{n-d} \in C^k(V)$ with

$$M \cap U = \{x \in \mathbb{R}^n \mid (x_{d+1}, \ldots, x_n) = g(x_1, \ldots, x_d)\}.$$

Notice that the sets and functions may depend on the point $p \in M$ we are looking at.

Proof. Lecture Notes, cry, optional □

We summarize some ideas.

Implicit: $M \cap U = \{x \in U \mid f(x) = 0\}, f : U \to \mathbb{R}^{n-d}$. For a plane, this may be $f(x, y, z) = ax + by + cz - h$. For a sphere, this may be $f(x, y, z) = x^2 + y^2 + z^2 - r^2$.

Parametric: $M \cap U = \{G(y) \mid y \in V\}, G : V \to \mathbb{R}^n$. For a plane, this may be $(s, t) \mapsto (sv_1 + tv_2 + p)$, where v_1, v_2 are two linearly independent vectors in the plane and p is a point in the plane. For a sphere, this may be $(\theta, \phi) \mapsto (r\cos(\theta)\cos(\phi), r\cos(\theta)\sin(\phi), r\sin(\theta))$. Here, $V = (0, 2\pi) \times (-\frac{\pi}{2}, \frac{\pi}{2})$.

Graph: $M \cap U = \{(y, g(y)) \mid y \in V \subset \mathbb{R}^d\}$. If we call this first part $G(y)$ we get a parametrization of the form $G(y) = (y, g(y))$. For a plane, this may be $z = -\frac{a}{c}x - \frac{b}{c}y + \frac{h}{c}$, and $x = -\frac{b}{a}y - \frac{c}{a}z + \frac{h}{a}$, so we get two different graphs. For a sphere, this may be $z = \pm\sqrt{r^2 - x^2 - y^2}$, together with $y = \pm\sqrt{r^2 - x^2 - z^2}$ and $x = \pm\sqrt{r^2 - y^2 - z^2}$, so we get 3 different graphs. All theese are required to cover the whole sphere, since we cannot write the whole sphere as a graph of a function.

Example 4.9: Using the implicit function definition

We want to show that the circle $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ is a 1-dimensional submanifold of $\mathbb{R}^2$. Consider $F : \mathbb{R}^2 \to \mathbb{R}, (x, y) \mapsto x^2 + y^2$. Then the jacobian $JF_{(x,y)} = (2x, 2y)$ has rank 1 for all $(x, y) \in S^1$. So $F^{-1}(\{1\}) = S^1$ is a 1-dimensional submanifold of $\mathbb{R}^2$.

Example 4.10: Using the parametric definition

Let $P := \{(x, t) \in \mathbb{R}^n \times \mathbb{R} : t = |x|^2\}$ be the n-dimensional paraboloid. Consider $f : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}, x \mapsto (x, |x|^2)$. Then $Df_x = \begin{pmatrix} I_n \\ 2x^T \end{pmatrix}$ has rank n for all $x \in \mathbb{R}^n$.

Furthermore, f is bijective since $(x, |x|^2) = (x', |x'|^2)$ implies $x = x'$. Furthermore since the differential exists, f is continuous. Furthermore, $f^{-1} : P \to \mathbb{R}^n, (x, t) \mapsto x$ is continuous as well.

Definition 4.11: Permutation of Coordinates

$P : \mathbb{R}^n \to \mathbb{R}^n$ is a PERMUTATION OF COORDINATES if

$$P(x_1, \ldots, x_n) = (x_{\sigma(1)}, \ldots, x_{\sigma(n)}) \text{ for some } \sigma \in S_n.$$

Definition 4.12: Tangent and Normal Vectors

Given $M \subset \mathbb{R}^n$, a d-dimensional submanifold of $\mathbb{R}^n$, then $\tau \in \mathbb{R}^n$ is TANGENT to M at $p \in M$ if $\exists p_k \in M, r_k > 0$ such that $\frac{p_k - p}{r_k} \to \tau$, where $p_k \to p$ and $r_k \to 0$ as $k \to \infty$.

$\nu \in \mathbb{R}^n$ is NORMAL to M at $p \in M$ if $\nu \cdot \tau = 0$ for all $\tau \in T_pM$, the set of tangent vectors to M at p. Lec 18

Proposition 4.13:

$\tau \in T_pM$ iff $\tau \in (D\psi)^{-1}_{\psi(p)}(E)$, where $E = \{y \in \mathbb{R}^n \mid y_{d+1} = \cdots = y_n = 0\}$.

In particular, $T_pM \subset \mathbb{R}^n$ is a d-dimensional vector subspace.

Proof. $\Rightarrow$) We compute

$$\lim_{k\to\infty} \underbrace{\frac{\psi(p_k) - \psi(p)}{r_k}}_{\in E} = \lim_{k\to\infty} \frac{D\psi_p(p_k - p) + o(|p_k - p|)}{r_k}$$
$$= \lim_{k\to\infty} D\psi_p \left(\frac{p_k - p}{r_k}\right) = D\psi_p(\tau).$$

Since E is closed, $D\psi_p(\tau) \in E$, so $\tau \in (D\psi)^{-1}_p(E)$.

$\Leftarrow$) Extra Material □

Also we have $T_pM = D\phi_p(\mathbb{R}^d)$ for any parametrization $\phi : V \to M \cap U$ with $\phi(q) = p$. Hence, T_pM is a d-dimensional vector subspace of $\mathbb{R}^n$.

Definition 4.14: Parametrized surface

Let $V \subset \mathbb{R}^2$ be open, $\phi : V \to \mathbb{R}^3$ a C^k map with $k \geq 1$ and $\operatorname{rank} D\phi_y = 2$ for all $y \in V$ and every $V' \subset V$ is open, then $\exists U' \subset \mathbb{R}^3$ open such that

$$\phi(V) \cap U' = \phi(V').$$

Watch Lecture

5 Multidimensional Integration

As a notation for the following section, we write for $X \subset \mathbb{R}^n$ and $\rho > 0, \alpha \in \mathbb{R}^n$,

$$\rho X + \alpha \coloneqq \{\rho x + \alpha \mid x \in X\}.$$

For example, $\{\alpha + [0,1)^n \mid \alpha \in \mathbb{Z}^n\}$ covers $\mathbb{R}^n$ without overlapping.

Definition 5.1: Dyadic Cubes

Given $p \in \mathbb{N}$ we say that $Q \subset \mathbb{R}^n$ is a DYADIC CUBE of length 2^{-p} if $\exists \alpha \in \mathbb{Z}^n$ such that $Q = 2^{-p}(\alpha + [0,1)^n)$. 2^{-p} is called the PIXEL SIZE.

$F \subset \mathbb{R}^n$ is a DYADIC SUBSET if it is a finite union of disjoint dyadic cubes, i.e. $\exists p \in \mathbb{N}$, $N \in \mathbb{N}$, and a map $\alpha : \{1, \dots, N\} \to \mathbb{Z}^n$ injective, such that

$$F = \bigcup_{i=1}^{N} 2^{-p}(\alpha(i) + [0,1)^n).$$

Definition 5.2: Volume of a Dyadic Subset

Let $F \subset \mathbb{R}^n$ be a dyadic set with $F = \bigcup_{i=1}^{N} 2^{-p}(\alpha(i) + [0,1)^n)$.

Then we define the VOLUME of F as

$$\mu(F) = N \cdot 2^{-pn}.$$

Note that $\mu(F)$ is well-defined, i.e. does not depend on the choice of p, N, α.

Proposition 5.3: Properties of μ

μ defined on dyadic set satisfies:

1) additive: F_1, F_2 disjoint: $\mu(F_1 \cup F_2) = \mu(F_1) + \mu(F_2)$.

2) normalisation: $\mu([0,1)^n) = 1$.

3) translation invariance: $\mu(F + \alpha) = \mu(F)$ for all $\alpha \in \mathbb{Z}^n$.

We observe that μ is the only map defined on dyadic sets that satisfies all these properties.

Definition 5.4: Outer / inner Volume

Let $E \subset \mathbb{R}^n$. We define the OUTER VOLUME of E as

$$\mu_{\text{out}}(E) = \inf\{\mu(f) \mid E \subset G, G \text{ diadic }\}.$$

We define the INNER VOLUME of E as

$$\mu_{\text{in}}(E) = \sup\{\mu(F) \mid F \subset E, F \text{ diadic }\}.$$

Definition 5.5: Jordan Measurable sets and volume

$E \subset \mathbb{R}^n$ bounded is JORDAN MEASURABLE if we have $\mu_{\text{out}}(E) = \mu_{\text{in}}(E)$.

This number is denoted $\mu(E) = \operatorname{vol}_n(E) \coloneqq \mu_{\text{in}}(E)$ and is called the VOLUME of E.

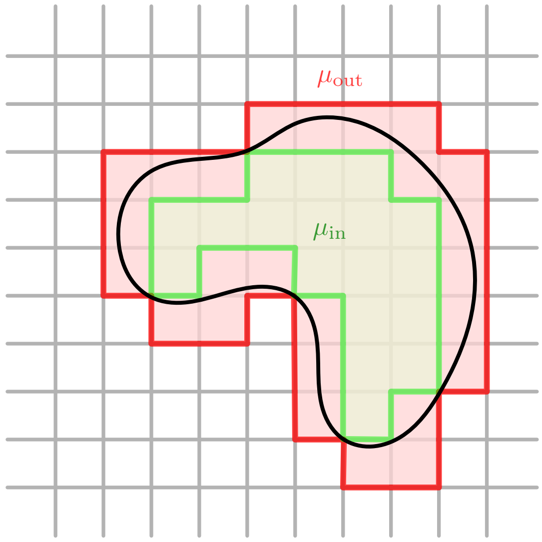

Figure 11: Approximation of the inner and outer volume of a set.

Example 5.6: Jordan measurable sets

The set $\{x \mid |x| < 1\} = B_1(0)$ is Jordan measurable.

The set $[a, b] \times [c, d] \subset \mathbb{R}^2$ is Jordan measurable.

The set $[0, 1) \cap \mathbb{Q}^n$ is not Jordan measurable.

Lemma 5.7:

1) μ_{out} is subadditive, i.e. for $E \subset \bigcup_{i=1}^N E_i$ we have

$$\mu_{\text{out}}(E) \leq \sum_{i=1}^N \mu_{\text{out}}(E_i).$$

2) μ_{in} is superadditive, i.e. for $E \supset \bigcup_{i=1}^N E_i$ with E_i disjoint, we have

$$\mu_{\text{in}}(E) \geq \sum_{i=1}^N \mu_{\text{in}}(E_i).$$

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Proof. We show this for the case $N = 2$. The general case is obtained by induction.

1) $\forall \varepsilon > 0$, $\exists G_1, G_2$ dyadic sets, such that $E_i \subset G_i$ and $\mu_{\text{out}}(E_i) \geq \mu(G_i) - \frac{\varepsilon}{2}$. Since $E_1 \cup E_2 \subset G_1 \cup G_2$, we have

$$\mu_{\text{out}}(E) \leq \mu(G_1 \cup G_2) \leq \mu(G_1) + \mu(G_2) \leq \sum_{i=1}^2 \mu_{\text{out}}(E_i) + \varepsilon.$$

Since ε was arbitrary, we get the desired result.

2) $\forall \varepsilon > 0$, $\exists F_1, F_2$ dyadic sets, such that $F_i \subset E_i$ and $\mu_{\text{in}}(E_i) \leq \mu(F_i) + \frac{\varepsilon}{2}$. Since $E_1 \cap E_2 = \varnothing$, we have $F_1 \cap F_2 = \varnothing$. Hence, $F_1 \cup F_2$ is a dyadic set and $F_1 \cup F_2 \subset E$. Thus,

$$\mu_{\text{in}}(E) \geq \mu(F_1 \cup F_2) = \mu(F_1) + \mu(F_2) \geq \sum_{i=1}^2 \mu_{\text{in}}(E_i) - \varepsilon.$$

Since ε was arbitrary, we get the desired result. $\square$

Lemma 5.8: Volume of boxes

$E = (a_1, b_1) \times \cdots \times (a_n, b_n) \subset \mathbb{R}^n$ with $a_i < b_i$ and $\overline{E}$ are Jordan measurable and

$$\operatorname{vol}_n(E) = \operatorname{vol}_n(\overline{E}) = \prod_{i=1}^n (b_i - a_i).$$

Proof. Let $p \in \mathbb{N}$ large. We say that

$$\underline{t} = \max\{2^{-p}k | k \in \mathbb{Z} \text{s.t.} 2^{-p}k < t\}$$
$$\overline{t} = \min\{2^{-p}k | k \in \mathbb{Z} \text{s.t.} 2^{-p}k > t\}.$$

$\forall t \in \mathbb{R}$, $\overline{t} - \underline{t} = 2^{-p}$ and $\underline{t} \leq t \leq \overline{t}$. Letting $p \to \infty$, we get $\underline{t}, \overline{t} \to t$.

Notice that $\overline{E} \subset \prod_{i=1}^n [\underline{a_i}, \overline{b_i})$ and $E \supset \prod_{i=1}^n [\overline{a_i}, \underline{b_i})$. Hence,

$$\prod_{i=1}^n (\underline{b_i} - \overline{a_i}) \leq \mu_{\text{in}}(E) \leq \mu_{\text{out}}(E) \leq \prod_{i=1}^n (\overline{b_i} - \underline{a_i}).$$

Letting $p \to \infty$, we get $\mu_{\text{in}}(E) = \mu_{\text{out}}(E) = \prod_{i=1}^n (b_i - a_i)$. $\square$

Proposition 5.9:

If E_1, E_2 are Jordan measurable, also $E_1 \cup E_2, E_1 \cap E_2, E_1 \setminus E_2$ are Jordan measurable. Moreover, μ is additive.

Proof. Step 1: If E_1, E_2 disjoint are Jordan measurable, then $E_1 \cup E_2$ is Jordan measurable and $\mu(E_1 \cup E_2) = \mu(E_1) + \mu(E_2)$.

We have that $\mu_{\text{out}}(E_i) = \mu_{\text{in}}(E_i)$. Hence, by subadditivity and superadditivity, we get

$$\mu_{\text{out}}(E_1 \cup E_2) \leq \mu_{\text{out}}(E_1) + \mu_{\text{out}}(E_2)$$
$$\mu_{\text{in}}(E_1) + \mu_{\text{in}}(E_2) \leq \mu_{\text{in}}(E_1 \cup E_2).$$

Since $\mu_{\text{out}}(E_i) = \mu_{\text{in}}(E_i)$, we get $\mu_{\text{out}}(E_1 \cup E_2) = \mu_{\text{in}}(E_1 \cup E_2)$, so $E_1 \cup E_2$ is Jordan measurable and $\mu(E_1 \cup E_2) = \mu(E_1) + \mu(E_2)$.

Step 2: Let E_1, E_2 be Jordan measurable. We show $E_1 \setminus E_2$ is Jordan measurable.

Since E_1, E_2 are Jordan measurable, $\forall \varepsilon > 0$, $\exists G_i, F_i$ dyadic such that $F_i \subset E_i \subset G_i$ and $\mu(G_i) - \mu(F_i) < \frac{\varepsilon}{2}$.

Then

$$\mu(\underbrace{F_1 \setminus G_2}_{:=F}) \leq \mu(E_1 \setminus E_2) \leq \mu(\underbrace{G_1 \setminus F_2}_{:=G}).$$

Notice, that

$$(G_1 \setminus F_2) \setminus (F_1 \setminus G_2) = (G_1 \setminus F_1) \cup (G_2 \setminus F_2).$$

By assumption,

$$\mu(G \setminus F) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

So the intersection is Jordan measurable.

Step 3: Let E_1, E_2 be Jordan measurable. We show $E_1 \cap E_2$ is Jordan measurable. We have $E_1 \cap E_2 = E_1 \setminus (E_1 \setminus E_2)$, so this follows from Step 2.

Also, since $E_1 \cup E_2 = E_1 \cap E_2 \cup (E_1 \setminus E_2) \cup (E_2 \setminus E_1)$, we have seen that all these sets are Jordan measurable and disjoint, so we are done. $\square$

Lemma 5.10: Sandwich Lemma

Given $E \subset \mathbb{R}^n$. If $\forall \varepsilon > 0$ there exists F, G Jordan measurable such that $F \subset E \subset G$ and $\mu(G) - \mu(F) < \varepsilon$, then E is Jordan measurable.

Proof. We have $\mu_{\text{out}}(E) \leq \mu_{\text{out}}(G) = \mu(G)$ and $\mu_{\text{in}}(E) \geq \mu_{\text{in}}(F) = \mu(F)$, so

$$\mu_{\text{out}}(E) - \mu_{\text{in}}(E) \leq \mu(G) - \mu(F) < \varepsilon.$$

Since ε was arbitrary, we get $\mu_{\text{out}}(E) = \mu_{\text{in}}(E)$, so E is Jordan measurable. $\square$

Definition 5.11: Jordan null set

$E \subset \mathbb{R}^n$ is called NULL if $\mu_{\text{out}}(E) = 0$.

In other words,

$$\forall \varepsilon > 0, \exists G \text{ dyadic such that } E \subset G \text{ and } \mu(G) < \varepsilon.$$

Theorem 5.12: Jordan Measurability Criterion

$E \subset \mathbb{R}^n$ is Jordan measurable iff E is bounded and ∂E is null.

To prove the theorem we will use the following lemma.

Lemma 5.13:

If $E \subset \mathbb{R}^n$ is Jordan measurable, then $E^0, \overline{E}$ are Jordan measurable with the same measure.

Proof. Since E is Jordan measurable, $\forall \varepsilon > 0$, $\exists F, G$ dyadic sets such that $F \subset E \subset G$ and $\mu(G) - \mu(F) < \varepsilon$.

From this, we get

$$F^0 \subset E^0 \subset \overline{E} \subset \overline{G}.$$

Since $F = \bigcup_{l=1}^{N} 2^{-p}(\alpha(l) + [0,1)^n)$, we have

$$F^0 \geq \bigcup_{l=1}^{N} 2^{-p}(\alpha(l) + (0,1)^n).$$

By lemma 5.8, $\mu(F^0) = \mu(F)$. Similarly, $\mu(\overline{G}) = \mu(G)$. Hence,

$$\mu(F^0) \leq \mu_{\text{in}}(E^0) \leq \mu_{\text{out}}(\overline{E}) \leq \mu(\overline{G}).$$

Since $\mu(G) - \mu(F) < \varepsilon$, we conclude by sandwich lemma (5.10). □

Proof. [Theorem 5.12] $\Rightarrow$) Let E be Jordan measurable. Thus E is bounded. Since $E^0, \overline{E}$ are Jordan measurable with the same measure, we have

$$\partial E = \overline{E} \setminus E^0 = 0.$$

$\Leftarrow$) Let E be bounded and ∂E null. Given $p \in \mathbb{N}$, let

$$F_p := \bigcup \{Q \text{ dyadic cube of size } 2^{-p} \mid Q \subset E^0\}$$

$$G_p := \bigcup \{Q \text{ dyadic cube of size } 2^{-p} \mid Q \cap \overline{E} \neq \varnothing\}.$$

By construction, $F_p \subset E^0 \subset \overline{E} \subset G_p$. Since the boundary has $\mu_{\text{out}}(\partial E) = 0$, for every $\varepsilon > 0$, $\exists H$ dyadic such that $\partial E \subset H$ and $\mu(H) < \varepsilon$.

Let p be the pixel size associated to H and notice, that

$$G_p \setminus F_p \subset H.$$

This implies, that $\mu(G_p) - \mu(F_p) \leq \mu(H) < \varepsilon$. Since ε was arbitrary, we conclude by sandwich lemma (5.10).

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□

Lemma 5.14: Lipschitz preserves null

Given $E \subset \mathbb{R}^n$ null and $f : E \to \mathbb{R}^n$ Lipschitz, then $f(E)$ is null as well.

In the following we will write

$$Q_p(a) := a + 2^{-p}[0,1)^n.$$

Proof. Since E is null, $\forall \varepsilon > 0$, $\exists G$ dyadic, i.e.

$$G = \bigcup_{l=1}^{N} Q_p(a_l),$$

such that $E \subset G$ and $\mu(G) = N \cdot 2^{-pn} < \varepsilon$.

Let $x_a \in Q_p(a) \cap E \neq \varnothing$. Then

$$|f(x) - f(x_a)| \leq L|x - x_a| \leq L\sqrt{n}2^{-p} \quad \forall x \in Q_p(a).$$

So actually,

$$f(x) \in f(x_a) + L\sqrt{n}2^{-p}[-1,1]^n \quad \forall x \in Q_p(a).$$

Hence,

$$f(E) \subset \bigcup_{l=1}^{N} f(Q_p(c)) \subset \bigcup_{l=1}^{N} f(x_a) + L\sqrt{n}2^{-p}[-1,1]^n.$$

But then

$$\mu_{\text{out}}(f(E)) \leq N \cdot (2L\sqrt{n}2^{-p})^n = (2L\sqrt{n})^n \cdot N \cdot 2^{-pn} < (2L\sqrt{n})^n \varepsilon.$$

Since ε was arbitrary, we get $\mu_{\text{out}}(f(E)) = 0$, so $f(E)$ is null. □

Lemma 5.15: Graphs of uniform continuity

Given $E \subset \mathbb{R}^{n-1}$ bounded, $f : E \to \mathbb{R}$ uniformly continuous, then the graph of f is null.

The graph of f is defined as $\Gamma_f = \{(x, f(x)) \mid x \in E\}$.

Proof. Let $\varepsilon > 0$. Since f is uniformly continuous, $\exists \delta > 0$ such that

$$|f(x) - f(y)| < \varepsilon \quad \forall x, y \in E \text{ with } |x - y| < \delta.$$

Since E is bounded, $\exists N_0 \in \mathbb{N}$ such that

$$E \subset [-N_0, N_0]^{n-1}.$$

So we can write

$$E \subset \bigcup_{l=1}^{N} Q_p(a_l) \quad \text{for } p \geq 0.$$

To cover our massiv cube, we find that $N \leq (2N_0 2^p)^{n-1}$. Without loss of generality, assume $Q_p(a_l) \cap E \neq \varnothing$ for all l. Let $x_l \in Q_p(a_l) \cap E$.

For $x \in Q_p(a_l) \cap E$, we have $|x - x_l| < \sqrt{n}2^{-p}$. Hence, choosing p large enough, we get $|f(x) - f(x_l)| < \varepsilon$.

Now,

$$\Gamma \subset \bigcup_{l=1}^{N} Q_p(x_l) \times [f(x_l) - \varepsilon, f(x_l) + \varepsilon].$$

So we compute the outer volume of Γ:

$$\mu_{\text{out}}(\Gamma) \leq \sum_{l=1}^{N} (2^{-p})^{n-1} \cdot 2\varepsilon = 2\varepsilon \cdot N \cdot 2^{-p(n-1)} \leq (2N_0)^{n-1} \cdot 2\varepsilon.$$

Since ε was arbitrary, we get $\mu_{\text{out}}(\Gamma) = 0$, so Γ is null. □

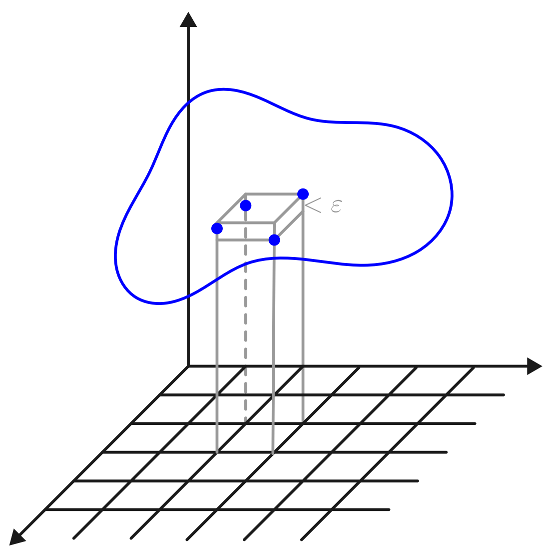

Figure 12: The graph of a uniformly continuous function is null.

Consider the linear map $x \mapsto \rho x + a$, where $\rho > 0$ and $a \in \mathbb{R}^n$.

Proposition 5.16: Translation and Dilation

Given $E \subset \mathbb{R}^n$ Jordan measurable, then $\rho E + a$ is Jordan measurable and $\mu(\rho E + a) = \rho^n \mu(E)$.

Proof. For a dyadic cube, $Q = a_0 + 2^{-p}[0,1)^n$, then $\rho Q + a$ is a box. So by lemma 5.8, $\mu(\rho Q + a) = \rho^n \mu(Q)$. From this, we immediately get that the same formula holds for dyadic sets as well.

Let E be Jordan measurable. $\forall \varepsilon > 0$, $\exists F, G$ dyadic such that $F \subset E \subset G$ and $\mu(G) - \mu(F) < \varepsilon$. Then since $\rho F + a \subset \rho E + a \subset \rho G + a$, we have

$$\mu(\rho F + a) \leq \mu_{\text{in}}(\rho E + a) \leq \mu_{\text{out}}(\rho E + a) \leq \mu(\rho G + a).$$

But by the first part of the proof, $\mu(\rho F + a) = \rho^n \mu(F)$ and $\mu(\rho G + a) = \rho^n \mu(G)$. Hence,

$$\rho^n \mu(G) - \rho^n \mu(F) = \rho^n(\mu(G) - \mu(F)) < \rho^n \varepsilon.$$

Since ε was arbitrary, we conclude by sandwich lemma (5.10). $\square$

Lemma 5.17:

Let $L : \mathbb{R}^n \to \mathbb{R}^n$ be linear and invertible. Then, given $E \subset \mathbb{R}^n$ Jordan measurable, $L(E)$ is Jordan measurable and

$$\mu(L(E)) = \mu(L([0,1)^n)) \cdot \mu(E).$$

Proof. Since E is Jordan measurable, ∂E is null by 5.12 Since linear Transformations are always Lipschitz, $L(\partial E) = \partial(L(E))$ is null as well. This implies that $L(E)$ is Jordan measurable.

Since E is Jordan measurable, $\forall \varepsilon > 0$, $\exists F, G$ dyadic such that

$$F \subset E \subset G \text{ and } \mu(G) - \mu(F) < \varepsilon.$$

This implies that $L(F) \subset L(E) \subset L(G)$. Let $A = L([0,1)^n)$. Since L is linear,

$$L(a + 2^{-p}[0,1)^n) = L(a) + 2^{-p}L([0,1)^n) = L(a) + 2^{-p}A.$$

This implies if F is a dyadic set, then

$$\mu(L(F)) = \underbrace{\mu(A)}_{\lambda(L)} \cdot \mu(F).$$

Similarly, $\mu(L(G)) = \mu(A) \cdot \mu(G)$. Hence, by sandwich lemma (5.10), we get $\mu(L(E)) = \mu(A) \cdot \mu(E)$. $\square$

Notice that the first part of the proof was needed to show that A is measurable.

Lemma 5.18: Special Stretching

Given $L : \mathbb{R}^n \to \mathbb{R}^n$ diagonal with entries $\lambda_1, \ldots, \lambda_n$ on the diagonal, then

$$\mu(L([0,1)^n) = \prod_{i=1}^{n} \lambda_i = \det(L).$$

Proof. This follows directly from the formula for boxes in lemma 5.8.

$$\lambda(L) = \mu(L([0,1)^n)) = \prod_{i=1}^{n} \lambda_i.$$

$\square$

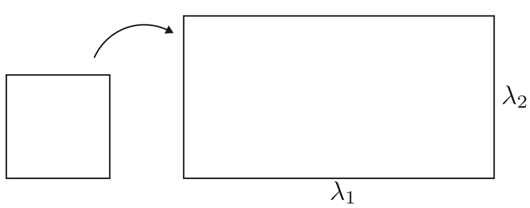

Figure 13: Stretching $[0,1)^2$ by a diagonal matrix.

Lemma 5.19: Decomposition

Given $L : \mathbb{R}^n \to \mathbb{R}^n$ linear and invertible, then $L = R_2 S R_1$, where $\det(R_i) > 0$ and $R_i^T R_i = I_n$ and S is diagonal with positive entries on the diagonal.

Proof. [Idea] $A = L^T L$ is a symmetric matrix. Thus by the spectral theorem, $L^T L = O^T D O$ for some orthogonal matrix O and some diagonal matrix D. $\square$

As a last observation, $B_1(0) \subset \mathbb{R}^n$ is Jordan measurable because $\partial B_1(0)$ is null as it can be covered by two graphs,

$$g_\pm(x') = \pm\sqrt{1 - |x'|^2},$$

where $g_\pm$ are uniformly continuous.

Theorem 5.20:

Given $L : \mathbb{R}^n \to \mathbb{R}^n$ linear. If E is Jordan measurable, then $L(E)$ is Jordan measurable and

$$\mu(L(E)) = |\det(L)| \cdot \mu(E).$$

Proof. We want to show $\lambda(L) = \det(L)$. Let $L = R_2 S R_1$ be the decomposition of L. Since R_i are orthogonal with positive determinant, they are rotations. So $\mu(RE) = \lambda(R) \cdot \mu(E) = \mu(E)$. But if we take this for the unit ball, we get $\lambda(R) = 1$. Hence, $\lambda(L) = \lambda(R_2)\lambda(S)\lambda(R_1) = \det(R_2)\det(S)\det(R_1) = \det(L)$. $\square$

5.1 The Riemann Integral

Lec 21

We now want to learn how to calculate the volume more easily.

Definition 5.21: Indicator Function

Let $A \subset \mathbb{R}^n$ be a set. The INDICATOR FUNCTION or CHARACTERISTIC FUNCTION of A is defined as

$$\mathbb{1}_A(x) := \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}.$$

Definition 5.22: Dyadic Step Function

A function $g : \mathbb{R}^n \to \mathbb{R}$ is a DYADIC STEP FUNCTION if

$$g(x) = \sum_{l=1}^{N} g_l \mathbb{1}_{Q_p(a_l)}(x).$$

where $g_l \in \mathbb{R}$, $a_l \in 2^{-p}\mathbb{Z}^n$ and $p \in \mathbb{N}$.

Definition 5.23: Integral of a Step Function

Let $g : \mathbb{R}^n \to \mathbb{R}$ be a dyadic step function. We define the INTEGRAL of g as

$$\int g(x) = \sum_{l=1}^{N} g_l \cdot \mu_n(Q_p(a_l)) = 2^{-pn} \sum_{l=1}^{N} g_l.$$

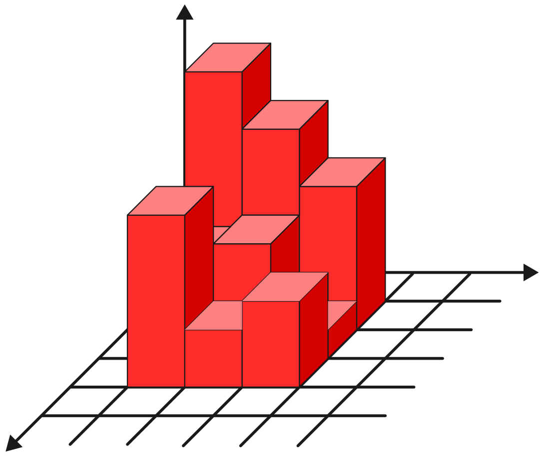

Figure 14: A step function.

Definition 5.24: Riemann Integrability

We say that $f : A \subset \mathbb{R}^n \to \mathbb{R}$ is RIEMANN INTEGRABLE over A if the upper and lower sums defined below coincide.

$$\mathcal{L}_A = \sup\left\{\int g \mid g \le f \cdot \mathbb{1}_A, g \text{ step function}\right\}$$
$$\mathcal{U}_A = \inf\left\{\int g \mid g \ge f \cdot \mathbb{1}_A, g \text{ step function}\right\}.$$

When a function is Riemann integrable, we write

$$\int_A f = \mathcal{L}_A = \mathcal{U}_A.$$

If we want to make explicit to which variable we are integrating, we write

$$\int_A f(x)\mathrm{d}x.$$

Where $\mathrm{d}x = \mathrm{d}x_1 \dots \mathrm{d}x_n$ which for the moment is just a notation.

Lemma 5.25:

Given $A \subset \mathbb{R}^n$ Jordan measurable and $c : A \to \mathbb{R}$ constant, then $c\mathbb{1}_A$ is Riemann integrable and

$$\int_A c = c\int_A \mathbb{1}_A = c\mu_n(A).$$

Proof. Exercise. □

Lemma 5.26:

Given $A \subset \mathbb{R}^n$ and $f_1, f_2 : A \to \mathbb{R}$ Riemann integrable, then for $c_1, c_2 \in \mathbb{R}$, $c_1 f_1 + c_2 f_2$ is Riemann integrable and

$$\int_A (c_1 f_1 + c_2 f_2) = c_1 \int_A f_1 + c_2 \int_A f_2.$$

Proof. Exercise.

Hint: Proof the formula for step functions first and then use the definition of Riemann integrability. □

Recall that given $f : X \to \mathbb{R}$, we have the positive and negative part of f defined as

$$f^+(x) = \max\{f(x), 0\} \quad \text{and} \quad f^-(x) = \max\{-f(x), 0\}.$$

With this, we can write $f = f^+ - f^-$ and $|f| = f^+ + f^-$.

Lemma 5.27:

$f : A \subset \mathbb{R}^n \to \mathbb{R}$ is Riemann integrable over A iff f^+, f^- are Riemann integrable over A. In particular,

$$\int_A f = \int_A f^+ - \int_A f^-.$$

Proof. Exercise. □

We now want to see how the Riemann integral is related to the volume of sets.

Lemma 5.28:

$f : A \subset \mathbb{R}^n \to [0, \infty)$ is Riemann integrable, iff

$$\Gamma_f = \{(x, x_{n+1}) | x \in A, 0 \le x_{n+1} \le f(x)\} \subset \mathbb{R}^{n+1},$$

is Jordan measurable. In this case,

$$\mu_{n+1}(\Gamma_f) = \int_A f.$$

Proof. Exercise. □

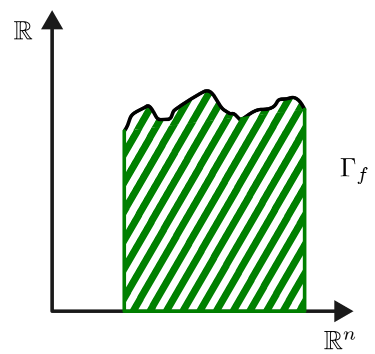

Figure 15: The hypergraph of a function.

Lemma 5.29:

Given $A \subset \mathbb{R}^n$ Jordan measurable and $f : A \to [0, \infty)$ uniformly continuous. Then, f is Riemann integrable over A.

Proof. f is Riemann integrable over A iff Γ_f is Jordan measurable. This is equivalent to $\mu_{n+1}(\partial\Gamma_f) = 0$. But

$$\partial\Gamma_f \subset A_1 \cup A_2 \cup A_3,$$

where

$$A_1 = \{(x, x_{n+1}) \mid x_{n+1} = f(x), x \in A\}$$
$$A_2 = \partial A \times [0, \sup_A f]$$
$$A_3 = A \times \{0\}.$$

Notice, that A_2 is null because ∂A is null (in $\mathbb{R}^n$)). Also, A_3 is null as it is a plane in $\mathbb{R}^{n+1}$. Finally, A_1 is null as it is the graph of a uniformly continuous function. Hence, $\partial\Gamma_f$ is null, so Γ_f is Jordan measurable, so f is Riemann integrable. □

5.2 Change of Variables

This is the final BOSS of integration.

Theorem 5.30: Change of Variables

Let $\Phi : U \subset \mathbb{R}^n \to V \subset \mathbb{R}^n$ be a C^1-diffeomorphism. Let $A \subset U$ be Jordan measurable, such that $\overline{A} \subset U$. Given $f : \overline{A} \to [0, \infty)$ continuous, then f is Riemann integrable over A and

$$\int_A f(x)\mathrm{d}x = \int_{\Phi(A)} \frac{f(\Phi^{-1}(y))}{|\det J\Phi(\Phi^{-1}(y))|}\mathrm{d}y.$$

The idea of the proof is depicted in the following figure.

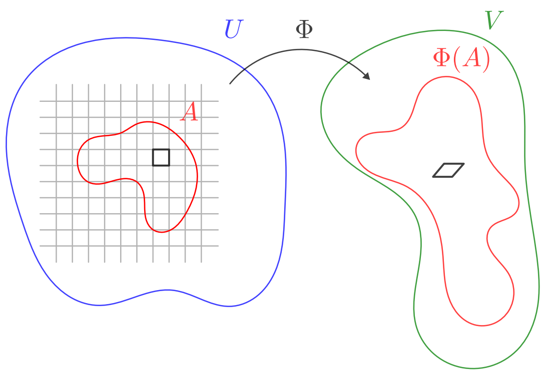

Figure 16: The idea of the change of variables formula.

Proof. 1. Since $\overline{A}$ is compact, f is uniformly continuous. By the previous lemma, f is Riemann integrable over A.

We have to show that $\Phi(A)$ is Jordan measurable and $\overline{\Phi(A)} \subset V$. Since Φ is differentiable, Φ is locally Lipschitz. Hence, $\Phi(A \cap B)$ is Jordan measurable for every ball B, centered on a point on the boundary of A. Since ∂A is null, we can cover it by countably many balls B_i. Rest of this part would be an exercise.

2. $\Phi(x_0 + Q_p) \approx D\Phi_{x_0}(Q_p)$. Given $Q_p = 2^{-p}[0, 1)^n$, define

$$Q_{p,\delta} := 2^{-p}[\delta, 1 - \delta)^n.$$

Then, $\forall \delta > 0$, $\exists p_\delta$ such that

$$D\Phi_{x_0}(Q_{p,\delta}) + \Phi(x_0) \subset \Phi(x_0 + Q_p).$$

If $y \in \Phi(x_0 + Q_p)$, then

$$\min_{x \in x_0 + Q_p} \frac{f(x)}{|\det J\Phi(x)|} \le \frac{f(\Phi^{-1}(y))}{|\det J\Phi(\Phi^{-1}(y))|}.$$

Then $\forall \varepsilon > 0$, we can find a step function g such that

$$\int_A f - \varepsilon \le \int_A g = \int_A \sum_{l=1}^{N} g_l \mathbb{1}_{Q_p(a_l)} \le \int_A f???????????.$$

□

We will revisit this a bit later in the course. For now, the
Lec 22 result is the important part, not the proof.

A better way to write the change of variables formula is given by letting $\Psi = \Phi^{-1}$ and writing

$$\int_A f(x)\mathrm{d}x = \int_{\Psi^{-1}(A)} f(\Psi(y))|\det J\Psi(y)|\mathrm{d}y.$$

This follows since by continuity, $\Phi \circ \Psi = \mathrm{id}$, so $J\Phi(\Psi(y)) \cdot J\Psi(y) = I_n$, so

$$|\det J\Phi(\Psi(y))| \cdot |\det J\Psi(y)| = 1.$$

This is similar to the substitution rule for one-dimensional integrals, where now, given $x = \Psi(y)$, we have

$$\mathrm{d}x = |\det J\Psi(y)|\mathrm{d}y.$$

Example 5.31: Area of the unit disk

The unit disk is given by $B_1 := \{x_1^2 + x_2^2 < 1\} \subset \mathbb{R}^2$. We want to compute $\mu_2(B_1)$.

To this extent we use polar coordinates

$$x_1 = y_1 \cos y_2$$
$$x_2 = y_1 \sin y_2.$$

So we get

$$\Psi(y_1, y_2) = \begin{pmatrix} y_1 \cos y_2 \\ y_1 \sin y_2 \end{pmatrix}.$$

The Domain of Ψ is $(0, 1) \times (0, 2\pi)$. Notice that $B_1 \neq \Psi((0, 1) \times (0, 2\pi))$ because the image of Ψ does not contain the positive part of the x_1-axis. However. This part has measure zero, for example since we can cover it with $\frac{1}{\varepsilon}$ boxes of size $\varepsilon \times \varepsilon$, so we can ignore it for the purpose of computing the area. Hence, we can compute

$$\begin{aligned}\mu_2(B_1) = \mu_2(\Psi(D)) &= \int_{\Psi(D)} 1\mathrm{d}x \\ &= \int_D |\det J\Psi(y)|\mathrm{d}y \\ &= \int_0^1 \int_0^{2\pi} y_1 \mathrm{d}y_2\mathrm{d}y_1 \\ &= 2\pi \int_0^1 y_1\mathrm{d}y_1 = \pi.\end{aligned}$$

Let us also compute $\mu_3(B_1) \subset \mathbb{R}^3$.

Example 5.32: Volume of the unit ball

We can use spherical coordinates to get

$$\Psi(y_1, y_2, y_3) = \begin{pmatrix} y_1 \cos y_2 \cos y_3 \\ y_1 \sin y_2 \cos y_3 \\ y_1 \sin y_3 \end{pmatrix}.$$

The Domain of Ψ is $(0, 1) \times (0, \pi) \times (0, 2\pi)$. Again, we can ignore the part of the image of Ψ that does not contain the positive part of the x_1-axis. Hence, we can compute

$$\begin{aligned}\mu_3(B_1) = \mu_3(\Psi(D)) &= \int_{\Psi(D)} 1\mathrm{d}x \\ &= \int_D |\det J\Psi(y)|\mathrm{d}y \\ &= \int_0^1 \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} y_1^2 \cos(y_3)\mathrm{d}y_3\mathrm{d}y_2\mathrm{d}y_1 \\ &= 2\pi \int_0^1 y_1^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(y_3)\mathrm{d}y_3\mathrm{d}y_1 \\ &= 4\pi \int_0^1 y_1^2\mathrm{d}y_1 = \frac{4\pi}{3}.\end{aligned}$$

To demonstrate how the above integrals were computed, we use a slicing idea. For this, consider first CAVALIERI'S PRINCIPLE. The idea is, that given a body $A \subset \mathbb{R}^3$, the volume of A can be computed by slicing A with planes

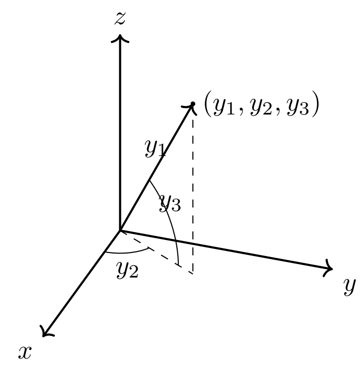

Figure 17: Spherical Coordinates

parallel to the x_1-axis and summing the areas of the slices, multiplied by their thickness.

Theorem 5.33: Cavalieri's Principle

For $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$, let $(x, y) \in \mathbb{R}^n$. Given $A \subset \mathbb{R}^n$ Jordan measurable, $A \subset (-C, C)^n$ for some $C > 0$, define the SLICES for $x \in (-C, C)^k$ as

$$A_x = \{y \in \mathbb{R}^{n-k} \mid (x, y) \in A\}.$$

Let also $\overline{\varphi}(x) = \mu_{n-k,\mathrm{out}}(A_x)$, $\underline{\varphi}(x) = \mu_{n-k,\mathrm{in}}(A_x)$.

Then, $\overline{\varphi}, \underline{\varphi} : [-C, C]^k \to \mathbb{R}$ are Riemann integrable and

$$\int_{(-C,C)^k} \underline{\varphi}(x)\mathrm{d}x = \mu_n(A) = \int_{(-C,C)^k} \overline{\varphi}(x)\mathrm{d}x.$$

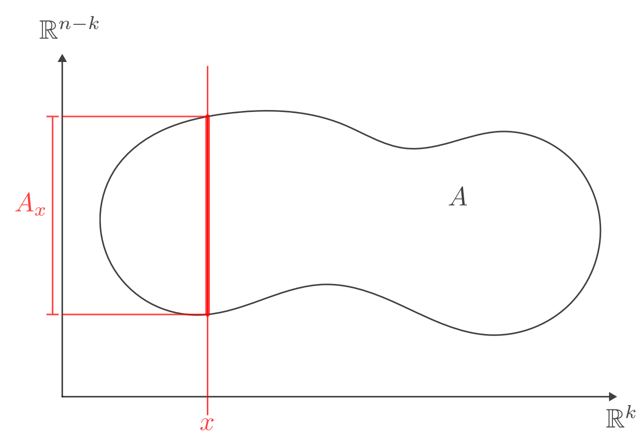

Figure 18: Cavalieri's principle.

Proof. Given dyadic cube (pixel 2^{-p}) in $\mathbb{R}^n$,

$$Q_p^n(z) = z + 2^{-p}[0, 1)^n.$$

Letting $z = (a, b)$, we have

$$Q_p^n(z) = Q_p^k(a) \times Q_p^{n-k}(b).$$

Now, given $\varepsilon > 0$, let $p \in \mathbb{N}$, F, G dyadic such that

$$F \subset A \subset G \text{ and } \mu(G) - \mu(F) < \varepsilon.$$

Now, $\forall x \in [-C, C]^k$, define

$$G_x = \{y \in \mathbb{R}^{n-k} \mid (x, y) \in G\}$$
$$F_x = \{y \in \mathbb{R}^{n-k} \mid (x, y) \in F\}.$$

We observe, that if $Q_p(a) \subset \mathbb{R}^k$ is a dyadic cube, then

$$F_x = F_{x'} \quad \forall x, x' \in Q_p(a)$$
$$G_x = G_{x'} \quad \forall x, x' \in Q_p(a).$$

Thus, $f(x) := \mu(F_x)$ and $g(x) := \mu(G_x)$ are dyadic step functions. Furthermore, $F_x \subset A_x \subset G_x$ implies that

$$\mu(F_x) \leq \underline{\varphi}(x) \leq \overline{\varphi}(x) \leq \mu(G_x).$$

Furthermore we have

$$\int f(x)\mathrm{d}x = 2^{-pn}\#\text{dyadic cubes in } F = \mu(F)$$
$$\int g(x)\mathrm{d}x = 2^{-pn}\#\text{dyadic cubes in } G = \mu(G).$$

So we get

$$\mu(F) = \int f(x)\mathrm{d}x \leq \underline{\int} \underline{\varphi}(x)\mathrm{d}x \leq \overline{\int} \overline{\varphi}(x)\mathrm{d}x \leq \int g(x)\mathrm{d}x = \mu(G).$$

But $\mu(G) - \mu(F) < \varepsilon$. Since ε was arbitrary, we conclude by sandwich lemma (5.10) that

$$\int \underline{\varphi}(x)\mathrm{d}x = \mu(A) = \int \overline{\varphi}(x)\mathrm{d}x.$$

□

Theorem 5.34: Fubini's Theorem

Let $A \subset \mathbb{R}^n$ be bounded. Write $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$ and let $(x, y) \in \mathbb{R}^n$ be the coordinates. Given $f : A \to \mathbb{R}$ Riemann integrable over A, then let

$$\tilde{f} = f \cdot \mathbb{1}_A = \begin{cases} f(x, y) & (x, y) \in A \\ 0 & x \notin A \end{cases}.$$

For $x \in \mathbb{R}^k$, define the marginals

$$\underline{\varphi}(x) = \underline{\int} \tilde{f}_x(y)\mathrm{d}y \quad \text{and} \quad \overline{\varphi}(x) = \overline{\int} \tilde{f}_x(y)\mathrm{d}y.$$

Where $\tilde{f}_x(y) = \tilde{f}(x, y)$. Then,

$$\int_A f = \int_{\mathbb{R}^k} \underline{\varphi}(x)\mathrm{d}x = \int_{\mathbb{R}^k} \overline{\varphi}(x)\mathrm{d}x.$$

The proof is mostly similar to the proof of Cavalieri's principle. As a reminder, $\underline{\varphi}(x)$ and $\overline{\varphi}(x)$ must not nessecairly be equal, if $\tilde{f}_x$ is not Riemann integrable. However, over the second integral, $\underline{\varphi}$ and $\overline{\varphi}$ are Riemann integrable and their integrals coincide. Lec 23

Corollary 5.35:

Given $K = [a_1, b_1] \times \cdots \times [a_n, b_n] \subset \mathbb{R}^n$ and $f : K \to \mathbb{R}$ continuous. Then, f is Riemann integrable over K and

$$\int_K f(x)\mathrm{d}x = \int_{a_1}^{b_1} \ldots \int_{a_n}^{b_n} f(x_1, \ldots, x_n)\mathrm{d}x_n \ldots \mathrm{d}x_1.$$

Proof. f is Riemann integrable according to Lemma 5.29.

We want to use Fubini's theorem with $k = 1$. Then, we have

$$\int_K f(x)\mathrm{d}x = \int_{a_1}^{b_1} \underline{\varphi}(x_1)\mathrm{d}x_1.$$

But $\underline{\varphi}(x_1) = \overline{\varphi}(x_1)$ for all $x_1 \in [a_1, b_1]$ because $\tilde{f}_{x_1}$ is continuous over $[a_2, b_2] \times \cdots \times [a_n, b_n]$. Hence, $\underline{\varphi}$ is continuous, so $\underline{\varphi}$ is Riemann integrable and

$$\int_K f(x)\mathrm{d}x = \int_{a_1}^{b_1} \int_{[a_2,b_2]\times\cdots\times[a_n,b_n]} f(x_1, x_2, \ldots, x_n)\mathrm{d}y\mathrm{d}x_1.$$

We now apply Fubini's theorem again to the inner integral, with $k = 1$ and get

$$\int_K f(x)\mathrm{d}x = \int_{a_1}^{b_1} \int_{a_2}^{b_2} \int f(x_1, x_2, x_3, \ldots, x_n)\mathrm{d}z\mathrm{d}x_2\mathrm{d}x_1.$$

Repeating this process $n-1$ times, we get the desired formula

$$\int_K f(x)\mathrm{d}x = \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} f(x_1, \ldots, x_n)\mathrm{d}x_n \ldots \mathrm{d}x_1.$$

$\square$

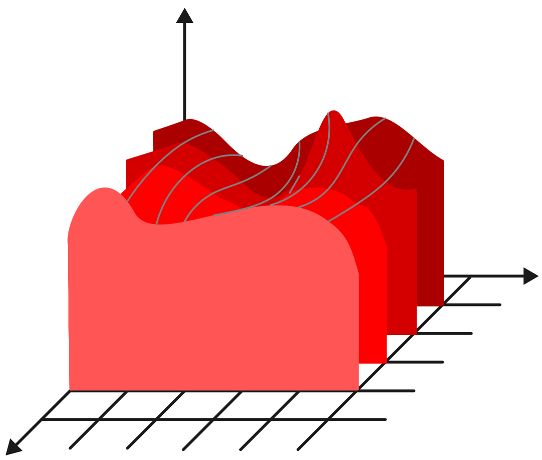

Figure 19: Fubini's theorem in two dimensions computes the area of each slice and sums them up.

Theorem 5.36: Differentiation under the integral

Given $K \subset \mathbb{R}^n$ compact and $f : K \times [a, b] \to \mathbb{R}$ continuous, $f = f(x, t)$, such that $\partial_t f$ exists and is continuous. Then, $\forall t_0 \in (a, b)$,

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t_0} \int_K f(x, t)\mathrm{d}x = \int_K \partial_t f(x, t_0)\mathrm{d}x.$$

Proof. By definition of the derivative, we have

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t_0} \int_K f(x, t)\mathrm{d}x = \lim_{h\to 0} \frac{1}{h}\left(\int_K f(x, t_0 + h)\mathrm{d}x - \int_K f(x, t_0)\mathrm{d}x\right).$$

By linearity of the integral, we can write

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t_0} \int_K f(x, t)\mathrm{d}x = \lim_{h\to 0} \int_K \frac{f(x, t_0 + h) - f(x, t_0)}{h}\mathrm{d}x.$$

By the mean value theorem, we know that there exists $\xi_x \in (t_0, t_0 + h)$ such that

$$\frac{f(x, t_0 + h) - f(x, t_0)}{h} = \partial_t f(x, \xi_x).$$

Since $\partial_t f$ is continuous on the compact set $K \times [a, b]$, $\partial_t f$ is uniformly continuous. Hence, $\forall \varepsilon > 0, \exists \delta > 0$ such that if $h < \delta$, or equivalently $|\xi_x - t_0| < \delta$, then

$$|\partial_t f(x, \xi_x) - \partial_t f(x, t_0)| < \varepsilon.$$

Thus, we have

$$\int_K \frac{f(x, t_0 + h) - f(x, t_0)}{h}\mathrm{d}x = \int_K \partial_t f(x, t_0) + E(x)\mathrm{d}x.$$

But the error is bounded by ε, so

$$\int_K \frac{f(x, t_0 + h) - f(x, t_0)}{h}\mathrm{d}x = \int_K \partial_t f(x, t_0)\mathrm{d}x + O(\varepsilon).$$

Letting $h \to 0$, we get

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t_0} \int_K f(x, t)\mathrm{d}x = \int_K \partial_t f(x, t_0)\mathrm{d}x.$$

$\square$

Definition 5.37: Improper Integral

Given $U \subset \mathbb{R}^n$ open and $f : U \to \mathbb{R}$ continuous, nonnegative. We define

$$\int_U f(x)\mathrm{d}x = \sup\left\{\int_K f \mid K \subset U, K\text{compact, JM}\right\}.$$

We observe, that whenever (K_l) with $K_0 \subset K_1 \subset \ldots$ such that $U = \bigcup_{l=0}^{\infty} K_l$, then

$$\int_U f = \lim_{l\to\infty} \int_{K_l} f.$$

Example 5.38: Gaussian Integral

We want to compute

$$I = \int_{-\infty}^{\infty} e^{-x^2}\mathrm{d}x.$$

To that extent, we compute I^2 as follows

$$\begin{aligned} I^2 &= \int_{-\infty}^{\infty} e^{-x^2}\mathrm{d}x \int_{-\infty}^{\infty} e^{-y^2}\mathrm{d}y \\ &= \int_{\mathbb{R}^2} e^{-(x^2+y^2)}\mathrm{d}x\mathrm{d}y. \end{aligned}$$

We now use polar coordinates to compute the last integral.

$$x = r\cos\theta, y = r\sin\theta.$$

So we get

$$\begin{aligned} I^2 &= \int_0^{2\pi} \int_0^{\infty} e^{-r^2} r\mathrm{d}r\mathrm{d}\theta \\ &= 2\pi \int_0^{\infty} e^{-r^2} r\mathrm{d}r \\ &= \pi \int_0^{\infty} e^{-u}\mathrm{d}u = \pi. \end{aligned}$$

Hence, $I = \sqrt{\pi}$.

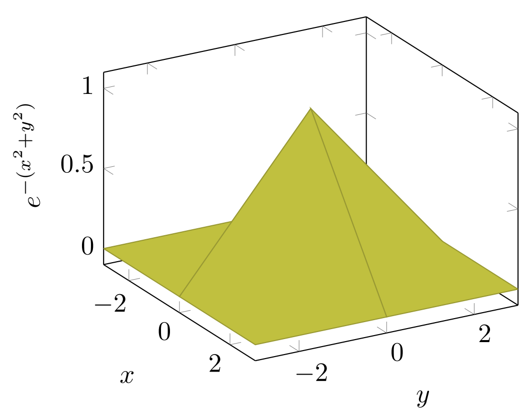

Figure 20: 3D plot of the Gaussian function.

Exercise 5.39:

Let a, b, c be vectors in $\mathbb{R}^3$, proof that

$$(a \times b) \cdot c = \det(a, b, c).$$

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Solution. We can simply compute all the entries of the cross product and the determinant and check that they are equal.

We now want to define integrals over d-dimensional submanifolds. To that extent, we first want to measure volumes over parametrized submanifolds. Let us start by $d = 1$. We call such a manifold a CURVE. We want to measure the LENGTH of a curve.

Recall that a parametrized 1-dimensional submanifold is given by $V \subset \mathbb{R}$ open and $\phi : V \to \mathbb{R}^n$ is injective and $\phi'(t) \neq 0$ for all $t \in V$.

Definition 5.40: Length

Given $[a, b] \subset V$, then we define the LENGTH of $\phi([a, b])$ as

$$L(\phi([a, b])) = \int_a^b |\phi'(t)| \mathrm{d}t.$$

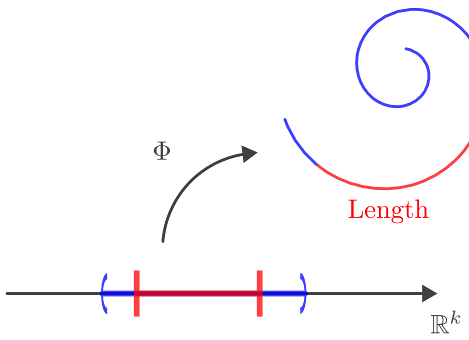

Figure 21: The length of a curve.

To check if this is a sensical definition, we want that length satisfies the following properties:

1. It should be independent of the chose parametrization. Given $\psi : [c, d] \to [a, b]$ a C^1-diffeomorphism, then we should have

$$L(\phi([a, b])) = L(\phi \circ \psi([c, d])).$$

As an exercise, we can check that this is indeed the case.

2. Invariance under euclidean isometries. I.e. for R orthogonal, $b \in \mathbb{R}^n$, we should have

$$L(\phi([a, b])) = L(R\phi([a, b]) + b).$$

3. Additivity: If $a < c < b$, then

$$L(\phi([a, b])) = L(\phi([a, c])) + L(\phi([c, b])).$$

4. Normalization: $[0, 1] \times \{0\}$ should have length 1.

Definition 5.41: d-volume on parametrized submanifolds

Given $V \subset \mathbb{R}^d$ open and $\phi : V \to \mathbb{R}^n$ a parametrized d-dimensional submanifold, and $E \subset V$ Jordan measurable with $\overline{E} \subset V$, we define the d-VOLUME as

$$\mathrm{vol}_d(\phi(E)) = \int_E \sqrt{\det((D\phi(x))^T D\phi(x))} \mathrm{d}x.$$

Here, $\det((D\phi(x))^T D\phi(x))$ is called the GRAM DETERMINANT.

The gram determinant is well-defined as if L is a $n \times d$ matrix with $d \leq n$ and rank d, then $L^T L$ is a $d \times d$ matrix of full rank.

For $d = 2$, we have the following lemma:

Lemma 5.42:

Given a 3×2 matrix (v, w), where v, w are column vectors in $\mathbb{R}^3$, then

$$\sqrt{\det((v, w)^T (v, w))} = |v \times w| = |v|^2 |w|^2 - (v \cdot w)^2.$$

In particular, if $\phi : V \subset \mathbb{R}^2 \to \mathbb{R}^3$ is a parametrized surface, then

$$A(\phi(E)) = \iint_E |\partial_1 \phi(x) \times \partial_2 \phi(x)| \mathrm{d}x_1 \mathrm{d}x_2.$$

Example 5.43: Area of the unit sphere

We can parametrize the unit sphere by

$$\Psi(\varphi, \theta) = \begin{pmatrix} \cos\varphi \cos\theta \\ \sin\varphi \cos\theta \\ \sin\theta \end{pmatrix}.$$

Then $A(\Psi((0, 2\pi) \times (-\frac{\pi}{2}, \frac{\pi}{2})))$ is given by

$$\begin{aligned} A &= \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{\det((D\Psi)^T D\Psi)} \mathrm{d}\theta \mathrm{d}\varphi \\ &= \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta \mathrm{d}\theta \mathrm{d}\varphi = 4\pi. \end{aligned}$$

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The properties we have defined for the length of curves, should also hold for the d-volume of d-dimensional submanifolds.

Proposition 5.44:

The four properties of length of curves also hold for the d-volume of d-dimensional submanifolds.

Proof. 1. Recall that for reparametrization, we have

$$\int_A \sqrt{\det(D\phi^T D\phi)} \mathrm{d}x = \int_{\psi^{-1}(A)} \sqrt{\det((D(\phi \circ \psi)^T D(\phi \circ \psi))} \mathrm{d}y.$$

Let ϕ_1 and ϕ_2 be two parametrizations of M so that $\phi_1 : V_1 \to M$ and $\phi_2 : V_2 \to M$ bijective.

Define $\psi = \phi_2^{-1} \circ \phi_1 : V_1 \to V_2$. Then, ψ is a C^1-diffeomorphism. Hence, $\phi_1 = \phi_2 \circ \psi$ is a reparametrization of ϕ_2.

Thus, by chain rule, we have

$$\begin{aligned} D(\phi \circ \psi)(y) &= D\phi(\psi(y)) \cdot D\psi(y) \\ D(\phi \circ \psi)(x)^T &= D\psi(x)^T \cdot D\phi(\psi(x))^T. \end{aligned}$$

Notice, that $D\psi(x)$ is a $d \times d$ matrix of rank d since ψ is a C^1-diffeomorphism. Thus, for the gram determinant, we have

$$\begin{aligned} &\det(D(\phi \circ \psi)^T D(\phi \circ \psi)) \\ &= \det(\underbrace{D\psi^T}_{d \times d} \underbrace{D(\phi) \circ \psi^T D\phi \circ \psi}_{d \times d} \underbrace{D\psi}_{d \times d}) \\ &= \det(D\psi^T) \det(D(\phi \circ \psi)^T D\phi \circ \psi) \det(D\psi) \\ &= \det(D\psi)^2 \det\left((D\phi) \circ \psi^T (D\phi) \circ \psi\right). \end{aligned}$$

We now apply change of variables with $x = \psi(y)$ to the LHS of the reparametrization formula to get

$$\begin{aligned}\int_A \sqrt{\det(D\phi^T D\phi)}\mathrm{d}x &= \int_{\psi^{-1}(A)} \sqrt{\det D\phi^T D\phi}(\psi(y))|\det D\psi(y)|\mathrm{d}y \\ &= \int_{\psi^{-1}(A)} \sqrt{\det D(\phi\circ\psi)^T D(\phi\circ\psi)(y)}\mathrm{d}y.\end{aligned}$$

2. Given R orthogonal, then $R^T R = I_n$. Hence, for the gram determinant,

$$\begin{aligned}&\det((R\phi)^T R\phi) \\ &= \det(\phi^T R^T R\phi) \\ &= \det(\phi^T\phi).\end{aligned}$$

Thus, we have

$$\begin{aligned}\mathrm{vol}_d(R\phi(E)+b) &= \int_E \sqrt{\det(D(R\phi)^T D(R\phi))}\mathrm{d}x \\ &= \int_E \sqrt{\det((R\phi)^T R\phi)}\mathrm{d}x \\ &= \int_E \sqrt{\det(\phi^T\phi)}\mathrm{d}x = \mathrm{vol}_d(\phi(E)).\end{aligned}$$

3. This follows immediately from the additivity of the integral.

4. Use the parametrization $\phi(x) = (x, 0, \dots, 0)$ and compute the gram determinant. Then, $\sqrt{\det((D\phi(x))^T D\phi(x))} = 1$, so

$$\mathrm{vol}_d(\phi([0,1]^d)) = \int_{[0,1]^d} 1\mathrm{d}x = 1.$$

□

As a notation, given L a $n\times d$ matrix with rand d, we will write

$$H = L\mathbb{R}^d = \{Lx \mid x\in\mathbb{R}^d\}.$$

H is a d-dimensional subspace of $\mathbb{R}^n$. For such matrices, $\exists R$ orthogonal, such that $RL = \begin{pmatrix}A\\0\end{pmatrix}$, where A is a $d\times d$ matrix of full rank.

Thus,

$$\sqrt{\det((RL)^T RL)} = \sqrt{\det(A^T A)} = \sqrt{\det(A)^2}.$$

From this we see that the Gram determinant is indeed in $\mathbb{R}$. Visually,

$$\sqrt{\det(A)^2} = \mathrm{vol}_d(A[0,1]^d) = \mathrm{vol}_d(L[0,1]^d).$$

Definition 5.45: Support of a function

Given $V\subset\mathbb{R}^m$ open, let $f : V\to\mathbb{R}^n$ be a continuous function. We define the SUPPORT of f as

$$\mathrm{spt}(f) = \overline{\{x\in V | f(x)\neq 0\}} \subset \overline{V}\subset\mathbb{R}^m.$$

We say that the f has COMPACT SUPPORT if $\mathrm{spt}(f)$ is compact (i.e. bounded)

Definition 5.46: Integral over parametrized submanifolds

Given $\phi : V\to\mathbb{R}^n$ a parametrized submanifold and $f:\phi(V)\to\mathbb{R}$ is continuous such that $\mathrm{spt}(f\circ\phi)$ is compact, and contained in V. Let $M = \phi(V)$, then we define the INTEGRAL of f over $\phi(V)$ as

$$\int_M f(p)\mathrm{d}\,\mathrm{vol}_M(p) = \int_V (f\circ\phi)\sqrt{\det((D\phi)^T D\phi)}\mathrm{d}x.$$

Observe that the integral exists since $f\circ\phi$ is continuous and has compact support, so $f\circ\phi$ is Riemann integrable. Lec 26

We want to analyze the gram determinant when $\phi(x') = (x' + g(x'))$. To this extent, let $V\subset\mathbb{R}^{n-1}$ open. Then,

$$D\phi = \begin{pmatrix}I_{n-1}\\ Dg(x')\end{pmatrix}.$$

Lemma 5.47:

Given $L = \begin{pmatrix}I_{n-1}\\ w^T\end{pmatrix}$, where $w\in\mathbb{R}^{n-1}$, then

$$\det(L^T L) = 1 + |w|^2.$$

Proof. Take R orthogonal such that $Rw = e_1|w|$. Hence,

$$LR^T = \begin{pmatrix}R^T\\ w^T R^T\end{pmatrix}.$$

Similarly,

$$RL^T = \begin{pmatrix}R & Rw\end{pmatrix}.$$

Thus we can compute

$$RL^T LR^T = \begin{pmatrix}R & Rw\end{pmatrix}\begin{pmatrix}R^T\\ w^T R^T\end{pmatrix} = I_{n-1} + |w|^2 e_1 e_1^T.$$

Hence, $\det(L^T L) = \det(RL^T LR^T) = 1 + |w|^2$. □

With this lemma, we get that the gram determinant of ϕ is given by

$$\sqrt{\det((D\phi)^T D\phi)} = \sqrt{1+|\nabla g(x')|^2}.$$

For the following, consider the following setup.

Let $V\subset\mathbb{R}^{n-1}$ open, $g : V\to\mathbb{R}$ C^1 and let

$$\Omega := B_r(0)\cap\{(x', x_n)\in\mathbb{R}^{n-1}\times\mathbb{R} \mid x_n < g(x')\}.$$

So Ω is the hypergraph intersected with a ball. Furthermore, let

$$\partial\Omega\subset\partial B_r(0)\cup\underbrace{\{(x', g(x')) \mid x_n = g(x')\}}_{M}.$$

Here we also require $O\in M$. Given $p\in M\cap B_r(0)$, put

$$\nu(p) = \nu(x') := \frac{(-\nabla g(x'), 1)}{\sqrt{1+|\nabla g(x')|^2}}.$$

Lemma 5.48:

ν is perpendicular to M at p, $|\nu(p)| = 1$ and ν is "pointing outwards" of Ω. So $p + h\nu(p)$ *not* $\in\Omega$ for $h>0$ small enough.

Proof. Let $\Phi(x') = (x', g(x'))$ be the parametrization of M. Then, if $p = \Phi(x')$, we have that the tangent space of M at p is spanned by $\partial_i\Phi$ for $i = 1,\dots,n-1$. We can compute

$$\partial_i\Phi(x') = e_i + \partial_i g(x')e_n.$$

As such, we have

$$\begin{aligned}\partial_i\Phi(x')\cdot\nu(x') &= \frac{e_i+\partial_i g(x')e_n}{\sqrt{1+|\nabla g(x')|^2}}\cdot(-\nabla g(x'), 1) \\ &= \frac{-\partial_i g(x')+\partial_i g(x')}{\sqrt{1+|\nabla g(x')|^2}} = 0.\end{aligned}$$

So ν is perpendicular to M at p. Furthermore, we have

$$|\nu(p)| = \frac{\sqrt{|\nabla g(x')|^2 + 1}}{\sqrt{1 + |\nabla g(x')|^2}} = 1.$$

Finally, we want to check that ν is pointing outwards of Ω. To that extent, let

$$q = (x', g(x')) + h(-\nabla g(x'), 1) = (x' - h\nabla g(x'), g(x') + h).$$

We claim that $y_n > g(y')$. For this, we taylor expand g with respect to h and get

$$g(y') = g(x' - h\nabla g(x')) = g(x') - h|\nabla g(x)|^2 < g(x') + h = y_n.$$

□

Definition 5.49: C^k function on a set

Given $A \subset \mathbb{R}^n$, we say that $f : A \to \mathbb{R}^m$ is C^k if there exists $U \supset A$ open and $\tilde{f} : U \to \mathbb{R}^m$ C^k such that $\tilde{f}|_A = f$.

Lemma 5.50: Divergence Theorem (miniboss)

In the setup above, let $f : \overline{\Omega} \to \mathbb{R}$ be C^1 and define $F = fe_n$. Assume that $\operatorname{spt}(f) \subset \overline{\Omega} \cap B_r(0)$. Then,

$$\int_\Omega \partial_n f \mathrm{d}x = \int_M f(p)\nu(p) \cdot e_n \mathrm{d}\operatorname{vol}_M(p).$$

Notice, that the vector field F is a vertical vector field (pointing in e_n). Furthermore, the support of F is contained in $\overline{\Omega} \cap B_r(0)$, so F can be nonzero on M, but must be zero on $\partial B_r(0)$ since the ball is open.

Proof. We have

$$\begin{aligned}
&\int_M f(p)\nu(p) \cdot e_n \mathrm{d}\operatorname{vol}_M(p)\\
&= \int_V f \circ \Phi \cdot \frac{(-\nabla g(x'), 1)}{\sqrt{1 + |\nabla g(x')|^2}} \cdot e_n \sqrt{1 + |\nabla g(x')|^2} \mathrm{d}x'\\
&= \int_V f \circ \Phi \mathrm{d}x' = \int_V f(x', g(x'))\mathrm{d}x'.
\end{aligned}$$

The left-hand side of the equation, since the function has compact support,

$$\int_\Omega \partial_n f \mathrm{d}x' = \int_{\{x_n < g(x')\}} \partial_n f(x', x_n).$$

By Fubini, we can write this as

$$\int_V \mathrm{d}x' \int_{-\infty}^{g(x')} \partial_n f(x', x_n)\mathrm{d}x_n.$$

But by the fundamental theorem of calculus, we have

$$\int_V \int_{-\infty}^{g(x')} \partial_n f(x', x_n)\mathrm{d}x_n \mathrm{d}x' = \int_V f(x', g(x'))\mathrm{d}x'.$$

□

Observe, that $\exists \varepsilon > 0, s \in (0, r)$ such that $\forall R$ orthogonal $n \times n$ matrix, $\|R - I_n\| < \varepsilon$. Then $RM \cap B_s(0)$ is still a
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graph $(y_n < \tilde{g}(y'))$ and $R\overline{\Omega} \cap B_s(0) \supset \operatorname{spt}(f \circ R^T)$.

Lemma 5.51:

In the setup above, let $f : \overline{\Omega} \to \mathbb{R}$ be C^1 with $\forall w \in \mathbb{R}^n$ we have

$$\int_\Omega \partial_w f \mathrm{d}x = \int_M f(p)w \cdot \nu(p)\mathrm{d}\operatorname{vol}_M(p).$$

Proof. Physicist variant: Let R be an orthogonal matrix such that $Re_n = w$. Then, we define

$$\begin{aligned}
\tilde{M} &= \{y_n = \tilde{g}(y')\}\\
\tilde{\Omega} &= \{y_n < \tilde{g}(y')\}\\
\tilde{f}(y) &= f(Rx).
\end{aligned}$$

We thus can apply lemma 5.50 to $\tilde{M}, \tilde{\Omega}, \tilde{f}$ and get

$$\int_{\tilde{\Omega}} \partial_{y_n} \tilde{f} \mathrm{d}y = \int_{\tilde{M}} \tilde{f} e_n \tilde{\nu} \mathrm{d}\operatorname{vol}_{\tilde{M}}.$$

Letting $w = w_n$, we get

$$\int_\Omega \partial_w f \mathrm{d}x = \int_M f(p)w \cdot \nu(p)\mathrm{d}\operatorname{vol}_M(p).$$

Notice that since this holds $\forall w$ such that $|w - e_n| < \varepsilon$, since the gradient is linear in w, we find

$$\partial_{aw+bv} f = \nabla f \cdot (aw + bv) = a\partial_w f + b\partial_v f.$$

So this equation is actually true for all $w \in B_\varepsilon(e_n) \cap S^{n-1}$. But the span of these vectors is $\mathbb{R}^n$, so the equation is actually true for all $w \in \mathbb{R}^n$. □

Definition 5.52: Divergence

Given $F = (F_1, \ldots, F_n) : \overline{\Omega} \to \mathbb{R}^n$, we define the DIVERGENCE of F as

$$\operatorname{div} F = \sum_{i=1}^n \partial_i F_i.$$

Theorem 5.53: Divergence Theorem, Local Case

In the setup above, let $F \in C^1(\overline{\Omega}, \mathbb{R}^n)$ such that $\operatorname{spt}(F) \subset \overline{\Omega} \cap B_r(0)$. Then,

$$\int_\Omega \operatorname{div} F \mathrm{d}x = \int_M F \cdot \nu \mathrm{d}\operatorname{vol}_M.$$

Proof. Using Lemma 5.51 with $w = e_i$ for $i = 1, \ldots, n$ and $f = F_i$, we get

$$\int_\Omega \partial_i F_i \mathrm{d}x = \int_M F_i(p) e_i \cdot \nu(p)\mathrm{d}\operatorname{vol}_M(p).$$

By linearity of the integral, we can sum over i to get

$$\int_\Omega \operatorname{div} F \mathrm{d}x = \int_M F(p) \cdot \nu(p)\mathrm{d}\operatorname{vol}_M(p).$$

□

We want to generalize the divergence theorem to more general domains. We have seen, that

$$\int_{M_p} f \mathrm{d}\operatorname{vol}_{M_p} = \int_{B_r(0)} f \circ \Phi \sqrt{\det((D\Phi)^T D\Phi)}\mathrm{d}x.$$

But how would we define $\int_M f \mathrm{d}\operatorname{vol}_M$ In general, there might not be a single parametrization of M. However, we can try and cut f into pieces, $f = \sum f_l$ where each f_l is supported on a parametrized piece M_{p_l} of M. Then, we can define

$$\int_M f \mathrm{d}\operatorname{vol}_M = \sum_l \int_{M_{p_l}} f_l \mathrm{d}\operatorname{vol}_{M_{p_l}}.$$

We now want to see how we can decompose f into pieces.

Lemma 5.54: Partition of Unity

Given $K \subset \mathbb{R}^n$ compact and $B_{p_l}(r_l)$ for $l = 1, \dots, N$ open balls, such that $K \subset \bigcup_{l=1}^N B_{p_l}(r_l)$, then $\exists U$ open such that $K \subset U$, N functions $\eta_1, \dots, \eta_N : \mathbb{R}^n \to [0, \infty)$ and one background function $\tilde{\eta} : \mathbb{R}^n \to [0, \infty)$ all C^∞ such that

$$1 = \sum_{l=1}^N \eta_l + \tilde{\eta}.$$

And

$$\operatorname{spt}(\eta_l) \subset B_{p_l}(r_l), \forall l = 1, \dots, N, \operatorname{spt}(\tilde{\eta}) \subset \mathbb{R}^n \setminus U.$$

Proof. Define

$$\xi(x) = \begin{cases} e^{-\frac{1}{1-|x|^2}} & B_1(0) \\ 0 & \text{otherwise} \end{cases}, \quad \tilde{\xi}(x) = \begin{cases} e^{-\frac{1}{|x|^2-1}} & \mathbb{R}^n \setminus B_1(0) \\ 0 & \text{otherwise} \end{cases}.$$

Both of these are C^∞ functions. Furthermore, $\operatorname{spt}(\xi) = \overline{B_1(0)}$.

For $r > 0, p \in \mathbb{R}^n$, define

$$\xi_{p,r}(x) := \xi\left(\frac{x-p}{r}\right).$$

Thus, $\operatorname{spt}(\xi_{p,r}) = \overline{B_r(p)}$.

Since K is compact and $K \subset \bigcup_{\theta \in (0,1)} \bigcup_{l=1}^N B_{\theta r_l}(p_l)$, there exists θ such that $K \subset \bigcup_{l=1}^N B_{\theta r_l}(p_l)$. Take

$$\xi_l(x) = \xi\left(\frac{x-p_l}{\theta^{\frac{1}{2}} r_l}\right), \quad \tilde{\xi}(x) = \tilde{\xi}\left(\frac{x-p_l}{\theta r_l}\right).$$

Also, let

$$\tilde{\xi}_0(x) = \prod_{i=1}^N \tilde{\xi}_{l(x)}.$$

Here, $\tilde{\xi}_0$ is 0 in $\bigcup_{l=1}^N B_{\theta r_l}(p_l) =: U$.

Define $S(x) := \sum_{l=1}^N \xi_l(x) + \tilde{\xi}_0(x)$. By construction, $S(x) > 0$ for all $x \in \mathbb{R}^n$.

Finally, take $\eta_l(x) = \frac{\xi_l(x)}{S(x)}$ and $\tilde{\eta}(x) = \frac{\tilde{\xi}_0(x)}{S(x)}$. Then, η_l and $\tilde{\eta}$ are C^∞ functions, and

$$1 = \frac{\sum_{l=1}^N \xi_l(x) + \tilde{\xi}_0(x)}{S(x)} = \sum_{l=1}^N \eta_l(x) + \tilde{\eta}(x).$$

Furthermore, $\operatorname{spt}(\eta_l) \subset B_{p_l}(r_l)$ and $\operatorname{spt}(\tilde{\eta}) \subset \mathbb{R}^n \setminus U$.

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□

Definition 5.55: C^k domain

Given $k \geq 1$, we say that $\Omega \subset \mathbb{R}^n$ open, bounded is a C^k DOMAIN if $\forall p \in \partial\Omega$, $\exists r > 0$ and F euclidean isometry such that $F(B_r(p)) = B_r(0)$ and $F(\Omega) \cap B_r(0)$ satisfies the assumptions of the setup above.

Theorem 5.56: Divergence Theorem

Let $\Omega \subset \mathbb{R}^n$ be a bounded C^1 domain and $G \in C^1(\overline{\Omega}, \mathbb{R}^n)$. Then,

$$\int_\Omega \operatorname{div} G \mathrm{d}x = \int_{\partial\Omega} G \cdot \nu \mathrm{d}\operatorname{vol}_{\partial\Omega}.$$

Proof. By definition of C^1 domain, $\forall p \in \partial\Omega$, $\exists r_p > 0$ and F_p euclidean isometry such that $F(B_r(p)) = B_r(0)$ and $F(\Omega) \cap B_r(0)$ satisfies the assumptions of the setup above. Since the boundary $\partial\Omega$ is compact (closed and bounded), $\exists p_1, \dots, p_N$ such that

$$\partial\Omega \subset \bigcup_{l=1}^N B_{r_l}(p_l).$$

Denote $F_l(x) = R_l x + a_l$ the euclidean isometry associated to p_l, where R_l is orthogonal and $a_l \in \mathbb{R}^n = -p_l$.

Let η_l be the corresponding partition of unity as in lemma 5.54, with $\tilde{\eta}$ denoting the background function. Define

$$G_l := G \cdot \eta_l \quad \tilde{G} = \begin{cases} G \cdot \tilde{\eta} & \text{in } \overline{\Omega} \\ 0 & \text{otherwise} \end{cases}.$$

Notice, that $\sum G_l + \tilde{G} = G$ in $\overline{\Omega}$. Furthermore,

$$\operatorname{spt}(G_l) \subset B_{r_l}(p_l), \quad \operatorname{spt}(\tilde{G}) \subset \Omega.$$

By construction, up to the isometry F_l, $\Omega \cap B_{r_l}(p_l)$ satisfies the assumptions of the setup above. Hence, we can apply the local divergence theorem (5.50) to G_l (actually to $G_l \circ F_l^{-1}$) and thus

$$\int_\Omega \operatorname{div} G_l \mathrm{d}x = \int_{\partial\Omega} G_l \cdot \nu \mathrm{d}\operatorname{vol}_{\partial\Omega}.$$

Moreover, since $\tilde{G} \in C^1(\overline{\Omega}, \mathbb{R}^n)$ and $\operatorname{spt}(\tilde{G}) \subset \Omega$, we claim that

$$\int_\Omega \operatorname{div} \tilde{G} \mathrm{d}x = 0.$$

To show this, we will compute $\int_\Omega \partial_i \tilde{G}_i \mathrm{d}x$ for $i = 1, \dots, n$.

$$\int_\Omega \tilde{G}_i(x + te_i) \mathrm{d}x = \int_{\Omega - te_i} \tilde{G}_i(x) \mathrm{d}x \quad \forall t \in (-\varepsilon, \varepsilon).$$

Thus, the function $t \mapsto \int_\Omega \tilde{G}_i(x + te_i) \mathrm{d}x$ is constant in t. By using differentiation under the integral sign, we get

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \tilde{G}_i(x + te_i) \mathrm{d}x = \int_\Omega \partial_i G_i(x) \mathrm{d}x = 0.$$

Summing over l, we get

$$\sum_{l=1}^N \int_\Omega \operatorname{div} G_l \mathrm{d}x + \int_\Omega \operatorname{div} \tilde{G} = \sum_{l=1}^N \int_{\partial\Omega} G_l \cdot \nu \mathrm{d}\operatorname{vol}_{\partial\Omega}$$

$$\int_\Omega \operatorname{div} G \mathrm{d}x = \int_{\partial\Omega} G \cdot \nu \mathrm{d}\operatorname{vol}_{\partial\Omega}.$$

□

Let us look at some applications of the divergence theorem.

Example 5.57: Archimedes principle

Given a body in water, the buoyant force is equal to the weight of the water displaced by the body. For this, we will use that the Force on some area A is given $F = A \cdot p \cdot \vec{\nu}$. Furthermore, the pressure p at some depth z is given by $p = \rho g z$, where ρ is the density of the water and g is the gravitational acceleration.

The force on the body is given by

$$F = \int_{\partial\Omega} p\nu \mathrm{d}\operatorname{vol}_{\partial\Omega} = g\rho \int_{\partial\Omega} z\nu \mathrm{d}\operatorname{vol}_{\partial\Omega}.$$

From this, we get

$$\vec{F} \cdot e_3 = \rho g \int_{\partial\Omega} z(\nu \cdot e_3) \mathrm{d}S = \rho g \int_\Omega (ze_3) \cdot \nu \mathrm{d}S.$$

Applying the divergence theorem, we get

$$\vec{F} \cdot e_3 = \rho g \int_\Omega \operatorname{div}(ze_3) \mathrm{d}x = \rho g \int_\Omega 1 \mathrm{d}x = \rho g \operatorname{vol}(\Omega).$$

Example 5.58: Surface of the sphere

Consider $B_1 \subset \mathbb{R}^n$. We want to compute the surface area of ∂B_1. For this, since $x = \nu(x)$ we can apply the divergence theorem

$$\begin{aligned}\mathrm{vol}_{n-1}(\partial B_1) &= \int_{\partial B_1} 1 \mathrm{d}\,\mathrm{vol}_{\partial B_1} \\ &= \int_{\partial B_1} x \cdot \nu \mathrm{d}\,\mathrm{vol}_{\partial B_1} \\ &= \int_{B_1} \mathrm{div} x \mathrm{d}x = n\,\mathrm{vol}(B_1).\end{aligned}$$

Example 5.59: Continuity Equation

The continuity equation states the local conservation of relevant quantities. For example, the density ρ.

The mass of fluid inside a controlled volume Ω is given by

$$m(t) = \int_\Omega \rho(t,x)\mathrm{d}x.$$

The continuity equation states that the change of mass in Ω is given by what flows in and out of Ω. Hence,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho \mathrm{d}x = \int_{\partial\Omega} \rho v \cdot \nu \mathrm{d}S.$$

Taking the derivative under the integral sign, and applying the divergence theorem, we get

$$\int_\Omega \partial_t \rho \mathrm{d}x = \int_\Omega \mathrm{div}(\rho v)\mathrm{d}x.$$

Since this holds for all Ω, we get the continuity equation

$$\partial_t \rho = \mathrm{div}(\rho v).$$

Example 5.60: Green Theorem

In $\mathbb{R}^2$, given $F \in C^1(\overline{\Omega}, \mathbb{R}^2)$, $\partial\Omega$ is a curve. Then,

$$\int_{\partial\Omega} \vec{F} \cdot \tau \mathrm{d}L = \int_\Omega \mathrm{rot}(\vec{F})\mathrm{d}x,$$

where $\mathrm{rot}(\vec{F}) = \partial_2 F_1 - \partial_1 F_2$ is the rotation of F.

Proof. Let $G = F^\perp = (-F_2, F_1)$. Then, $\mathrm{div}G = \mathrm{rot}F$. Furthermore, applying the divergence theorem to G, we get

$$\int_{\partial\Omega} G \cdot \nu \mathrm{d}L = \int_\Omega \mathrm{div}G \mathrm{d}x.$$

Notice that $G \cdot \nu = -F \cdot \tau$. Hence, we get

$$\int_{\partial\Omega} -F \cdot \tau \mathrm{d}L = \int_\Omega -\mathrm{rot}F \mathrm{d}x.$$

□

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Exercise 5.61:

Show the following identity for $h \in C^1(\overline{\Omega})$:

$$\int_\Omega \partial_i h \mathrm{d}x = \int_{\partial\Omega} h\nu_i \mathrm{d}\,\mathrm{vol}_{n-1}.$$

Exercise 5.62: Integration by parts

Show that for $g, h \in C^1(\overline{\Omega})$,

$$\int_\Omega \partial_i g h \mathrm{d}x = -\int_\Omega g \partial_i h \mathrm{d}x + \int_{\partial\Omega} g h \nu_i \mathrm{d}\,\mathrm{vol}_{n-1}.$$

Exercise 5.63:

Let $f \in C^1(\overline{\Omega}, \mathbb{R}^n)$ and $g \in C^1(\overline{\Omega})$. Show that

$$\int_\Omega \mathrm{div} f g = -\int_\Omega f \cdot \nabla g \mathrm{d}x + \int_{\partial\Omega} f g \cdot \nu \mathrm{d}\,\mathrm{vol}_{n-1}.$$

Hint: Show that $\mathrm{div}(fg) = \mathrm{div} f g + f \cdot \nabla g$

Warning: The divergence theorem also holds true in most "reasonable" domains. For example, a square is not a C^1 domain, but the divergence theorem still holds.

Exercise 5.64:

Prove the divergence theorem for $\Omega = [a,b] \times [c,d] \subset \mathbb{R}^2$ or any box in $\mathbb{R}^n$.

5.3 Differential Forms

Recall from linear algebra, that given a vector space V, we defined the dual space $V^* = \{\text{linear maps } V \to \mathbb{R}\}$. For us, $V = \mathbb{R}^n$, so

$$\mathbb{R}^{n*} = \mathbb{R}^n_{\text{row}} = \{\text{row vectors}\}.$$

Elements from $\mathbb{R}^n$ are called VECTORS, while elements from $\mathbb{R}^{n*}$ are called COVECTORS.

Definition 5.65: 1-Form

Given $U \subset \mathbb{R}^n$ open, a map α assigning to each point $x \in U$ a covector

$$x \mapsto (\alpha_1(x), \ldots, \alpha_n(x)) = \alpha_x.$$

is called a 1-FORM. (C^k 1-form if $\alpha_i \in C^k(U)\ \forall i$)

Example 5.66:

$F \in C^1(U, \mathbb{R}^n)$ is a (C^k) vector field. We can associate to F a 1-Form $\alpha = \alpha_F$ by $\alpha_x = F(x)^T$.

Example 5.67:

Given $f \in C^2(U, \mathbb{R})$, the map

$$x \in U \mapsto Df_x.$$

is a 1-form.

Definition 5.68:

Given $f \in C^\infty(U)$, we define

$$df : U \to \mathbb{R}^{n*}, \quad df_x = Df_x.$$

Definition 5.69: Line integral

Given $\alpha : U \to \mathbb{R}^{n*}$, C^1 is a 1-form, then we define the integral of α over a curve $\gamma : [a, b] \to U$ as

$$\int_\gamma \alpha := \int_a^b \alpha_{\gamma(t)}(\gamma'(t))\mathrm{d}t = \int_a^b \alpha(\gamma(t)) \cdot \gamma'(t)\mathrm{d}t.$$

Lemma 5.70:

Givne $\psi : [c, d] \to [a, b]$ a C^1 diffeomorphism, such that $\psi' > 0$.

$$\int_\gamma \alpha = \int_{\gamma\circ\psi} \alpha.$$

Proof. We have

$$\int_\gamma \alpha = \int_a^b \alpha(\gamma(t)) \cdot \gamma'(t)\mathrm{d}t.$$

On the other hand,

$$\int_{\gamma\circ\psi} \alpha = \int_c^d \alpha(\gamma(\psi(t))) \cdot (\gamma \circ \psi)'(t)\mathrm{d}t.$$

Applying the (1-dimensional) chain rule, $(\gamma \circ \psi)'(t) = \gamma'(\psi(t))\psi'(t)$, we get

$$\begin{aligned}\int_{\gamma\circ\psi} \alpha &= \int_c^d \alpha(\gamma(\psi(t))) \cdot \gamma'(\psi(t))\psi'(t)\mathrm{d}t \\ &= \int_c^d \alpha(\gamma(\psi(t))) \cdot (\gamma'(\psi(s)))\psi'(s)\mathrm{d}s \\ &= \int_a^b \alpha(\gamma(t)) \cdot \gamma'(t)\mathrm{d}t.\end{aligned}$$

□

We observe, that to each vector field F, associating $\alpha = F^T$, we have

$$\int_\gamma \alpha = \int_\gamma F^T = \int_a^b F(\gamma(t)) \cdot \gamma'(t)\mathrm{d}t.$$

This is exactly the classic line integral from physics.

Definition 5.71: Conservative vector fields

Given $U \subset \mathbb{R}^n$ connected, open, we say that $F : U \to \mathbb{R}^n, C^1$ is CONSERVATIVE if $\exists f : U \to \mathbb{R}$, C^2 such that $\nabla f = F$. f is called the POTENTIAL of F.

Equivalently, $\alpha : U \to \mathbb{R}^{n*}$, C^1 is EXACT if $\exists f : U \to \mathbb{R}$, C^2 such that $df = \alpha$.

Lemma 5.72:

If α is an exact 1-form, then $\int_\gamma \alpha$ only depends on the endpoints of γ. In particular, if $\alpha = df$,

$$\int_\gamma \alpha = f(\gamma(b)) - f(\gamma(a)).$$

Proof. We compute

$$\begin{aligned}\int_\gamma \alpha &= \int_a^b \alpha_{\gamma(t)}(\gamma'(t))\mathrm{d}t = \int_a^b df_{\gamma(t)}(\gamma'(t))\mathrm{d}t \\ &= \int_a^b Df_{\gamma(t)} \cdot \gamma'(t)\mathrm{d}t = \int_a^b (f \circ \gamma)'(t)\mathrm{d}t \\ &= f(\gamma(b)) - f(\gamma(a)).\end{aligned}$$

□

Lemma 5.73:

Given $\alpha : U \to \mathbb{R}^{n*}$ is exact, then setting $\alpha_x = (\alpha_1(x), \ldots, \alpha_n(x))$, we have

$$\partial_j\alpha_i = \partial_i\alpha_j, \forall i, j = 1, \ldots, n.$$

Proof. If $\alpha = \mathrm{d}f$, then

$$\alpha = (\partial_1 f, \ldots, \partial_n f).$$

But then, by Schwarz's theorem, we have

$$\partial_j\alpha_i = \partial_j\partial_i f = \partial_i\partial_j f = \partial_i\alpha_j.$$

□

Lemma 5.74:

Assume that $U \subset \mathbb{R}^n$ is convex and let $\alpha : U \to \mathbb{R}^{n*}$ satisfying integrability conditions. Then, $\exists f : C^2(U)$ such that $\mathrm{d}f = \alpha$.

Proof. We will do the proof for a box in $\mathbb{R}^2$. Let $U = (a, b) \times (c, d)$. Fix $(x_0, y_0) \in U$. We will define f as

$$\begin{aligned}f(x, y) &= \int_{x_0}^x \alpha_{(t,y_0)} \cdot e_1\mathrm{d}t + \int_{y_0}^y \alpha_{(x,s)} \cdot e_2\mathrm{d}s \\ &= \int_{x_0}^x \alpha_1(t, y_0)\mathrm{d}t + \int_{y_0}^y \alpha_2(x, s)\mathrm{d}s.\end{aligned}$$

We claim that $Df = \alpha$ or $\nabla f = \alpha^T$. Indeed, by fundamental theorem of calculus, and differentiation under the integral sign, we have

$$\partial_x f = \alpha_1(x, y_0) + \int_{y_0}^y \partial_x\alpha_2(x, s)\mathrm{d}s.$$

Similarly,

$$\partial_y f = \int_{x_0}^x \partial_y\alpha_1(t, y_0)\mathrm{d}t + \alpha_2(x, y).$$

By integrability conditions, $\partial_x\alpha_2 = \partial_y\alpha_1$, so

$$\begin{aligned}&\partial_x f = \alpha_1(x, y_0) + \alpha_1(x, y) - \alpha_1(x, y_0) = \alpha_1(x, y) \\ &\partial_y f = \alpha_2(x, y).\end{aligned}$$

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How can we generalize line integrals to higher dimensions? For this, we ask which quantities integrated over parametrized submanifolds of dimension k are invariant under orientation-preserving reparametrizations. So we want to find $\alpha_p(T) = \alpha(p; T)$, where $p \in U$ and T is a $n \times k$ matrix.

Let $\Phi : V \to \mathbb{R}^n$ be a parametrized submanifold with $M = \Phi(V)$. We define

$$\int_M \alpha = \int_V \alpha_{\Phi(x)}(D\Phi_x)\mathrm{d}x.$$

Here, the columns of $T = D\Phi_x$ are the tangent vectors to M at $\Phi(x)$, and span the tangent space $T_{\Phi(x)}M$.

Properties we want to have:

$$\alpha(p; TS) = \alpha(p; T) \cdot \det S.$$

Definition 5.75: k form in $\mathbb{R}^n$

A map $p \in U \subset \mathbb{R}^n$ assigns $\alpha_p = \alpha_p(T)$,

$$\alpha_p : \mathbb{R}^{n\times k} \to \mathbb{R}.$$

is called a k-FORM if α_p satisfies the following properties:

1. $\alpha_p(TS) = \alpha_p(T) \cdot \det S$
2. α_p is multilinear (linear in each column of T)

Example 5.76:

If $n = k$, then $\alpha(p;T) = f(p) \cdot \det T$ for some $f : U \to \mathbb{R}$ is a n-form.

Fix A, $k \times n$ matrix. And let $\alpha_A(T) = \det(AT)$. Then, $\alpha_A(p;TS) = \det(ATS) = \det(AT) \cdot \det S = \alpha_A(p;T) \cdot \det S$ is a k-form.

In fact, any map satisfying the properties is a linear combination of maps of the form α_A for some A.

One can show, that if α is a k-form, then

$$\alpha(T) = \sum_{1<i_1<\cdots<i_k<n} ci_1, \ldots, i_k(p) \cdot \det(E^T_{i_1,\ldots,i_k} T).$$

To simplify notation, we will define

$$\mathcal{I}_k = \{(i_1, \ldots, i_k) : 1 \le i_1 < \cdots < i_k \le n\}.$$

By combinatorics, $|\mathcal{I}_k| = \binom{n}{k}$.

Proposition 5.77:

α_p is invariant under orientation-preserving reparametrizations.

Proof. Let $\Psi : W \to V$ be a C^1 diffeomorphism with $\det D\Psi > 0$. Then,

$$\begin{aligned}
&\int_W \alpha(\Phi \circ \Psi(y); D\Phi_{\Psi(y)} D\Psi_y)\mathrm{d}y \\
&= \int_W \alpha(\Phi \circ \Psi(y); D\Phi(\Psi(y) \cdot D\Psi_y)\mathrm{d}y \\
&= \int_W \alpha(\Phi \circ \Psi(y); D\Phi_{\Psi(y)}) \cdot \det D\Psi_y \mathrm{d}y
\end{aligned}$$

.

By applying the change of variables formula, we get

$$\int_W \alpha(\Phi \circ \Psi(y); D\Phi_{\Psi(y)}) \cdot \det D\Psi_y \mathrm{d}y = \int_V \alpha(\Phi(x); D\Phi_x)\mathrm{d}x.$$

□

On k-forms, we can define the wedge product for fixed p and $p \to \alpha_p$.

Definition 5.78: Wedge product

Let α be a k-form and β be a m-form. If $k+m \le n$, and

$$\alpha(T) = \det(AT) \quad \beta(S) = \det(BS),$$

then we define for $X \in \mathbb{R}^{n\times(k+m)}$,

$$\alpha \wedge \beta(X) = \det\left(\begin{pmatrix} A \\ B \end{pmatrix} X\right).$$

For the general definition, we extend the definition in bilinear fashion.

Example 5.79:

Consider $x \mapsto \alpha_x = (0, x_2, x_3 - 2x_1)$ and $x \mapsto \beta_x = (1, x_3, \cos(x_2))$. Then $\alpha \wedge \beta$ is a 2-form, and

$$(\alpha \wedge \beta)_x(T) = \det \begin{pmatrix} 0 & x_2 & x_3 - 2x_1 \\ 1 & x_3 & \cos(x_2) \end{pmatrix} \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \\ v_{31} & v_{32} \end{pmatrix}.$$

Definition 5.80: Convention

0-Forms in $U \subset \mathbb{R}^n$ open are C^∞ functions.

k-forms in $U \subset \mathbb{R}^n$ open are

$$\sum_{I\in\mathcal{I}_k} c_I(x) \det(e_I^T T) = \alpha_x(T).$$

$k > n$-forms are 0.

Notice that using this, T is essentially k vectors of $\mathbb{R}^n$. So, we can think of a k-form as a function that takes k vectors and gives a number, and is linear in each vector, and changes sign if we swap two vectors.

Definition 5.81: Differential

Let $0 \le k \le n$ and $\alpha \in \Omega^k(U)$ be a k-form such that

$$\alpha_x(T) = \sum_{I\in\mathcal{I}_k} c_I(x) \underbrace{\det(e_I^T T)}_{:=w_I}.$$

We define the DIFFERENTIAL of α as the $(k+1)$-form

$$\beta := \mathrm{d}\alpha, \quad \beta_x(S) = \sum_I \mathrm{d}c_I(x) \wedge w_I(S).$$

Remark 5.82:

In $\mathbb{R}^n$, then coordinate functions $f(x) = x_i$ have differentials

$$\mathrm{d}x_i = \mathrm{d}f_x = Df_x = e_i^T.$$

Since any k-form is of the form $\sum c_I(x)w_I$, we can write

$$\omega_i = \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k} =: \mathrm{d}x_I.$$

So we can write $\alpha = \sum c_I(x)\mathrm{d}x_I$.

Exercise 5.83:

Given $\Psi : V \to U$ a C^∞ diffeomorphism, where $\Psi = (\Psi_1, \ldots, \Psi_n)^T$. Then, show that

$$\mathrm{d}\Psi_1 \wedge \cdots \wedge \mathrm{d}\Psi_n = \det D\Psi \cdot \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n.$$

Lemma 5.84:

$\forall \alpha \in \Omega^k(U)$, $\mathrm{d}(\mathrm{d}\alpha) = 0 \in \Omega^{k+2}(U)$.

$$d^2 = 0.$$

Proof. $\mathrm{d}\mathrm{d}(f(x)w_I) = \mathrm{d}fw_I = \sum_{i=1}^n \partial_i f \mathrm{d}x_i \wedge w_I$.

$$\mathrm{d}(\mathrm{d}f) = \sum_{i=1}^n \sum_{j=1}^n \partial_j \partial_i f \mathrm{d}x_j \wedge \mathrm{d}x_i = 0.$$

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□

We now want to see why differential forms are useful.

Lemma 5.85: Leibniz Rule

Given $\alpha \in \Omega^k(U)$ and $\beta \in \Omega^l(U)$, then

$$\mathrm{d}(\alpha \wedge \beta) = \mathrm{d}\alpha \wedge \beta + (-1)^k \alpha \wedge \mathrm{d}\beta.$$

Proof. We denote $e_I^*(A) := \det(e_I^T A)$. Then, we can write

$$\alpha = \sum_{I \in \mathcal{I}_k} \alpha_I e_I^*..$$

Recall that $\alpha_x(A) = \sum_{I \in \mathcal{I}_k} \alpha_I(x) e_I^*(A)$. By definition:

$$\mathrm{d}\alpha = \sum_{I \in \mathcal{I}_k} \mathrm{d}\alpha_I \wedge e_I^*, \quad \mathrm{d}\beta = \sum_{J \in \mathcal{I}_l} \mathrm{d}\beta_J \wedge e_J^*.$$

Furthermore, by definition of wedge product, we have

$$\alpha \wedge \beta = \sum_{I \in \mathcal{I}_k} \sum_{J \in \mathcal{I}_l} \alpha_I \beta_J e_I^* \wedge e_J^*.$$

Now, we compute

$$\begin{aligned} \mathrm{d}(\alpha \wedge \beta) &= \sum_{I \in \mathcal{I}_k} \sum_{J \in \mathcal{I}_l} \mathrm{d}(\alpha_I \beta_J) \wedge e_I^* \wedge e_J^* \\ &= \sum_{I \in \mathcal{I}_k} \sum_{J \in \mathcal{I}_l} (\mathrm{d}\alpha_I \beta_J + \alpha_I \mathrm{d}\beta_J) \wedge e_I^* \wedge e_J^* \\ &= \sum_{I \in \mathcal{I}_k} \sum_{J \in \mathcal{I}_l} \mathrm{d}\alpha_I \beta_J e_I^* \wedge e_J^* + \sum_{I \in \mathcal{I}_k} \sum_{J \in \mathcal{I}_l} \alpha_I \mathrm{d}\beta_J e_I^* \wedge e_J^* \end{aligned}$$

.

However, computing the other side, we get

$$\mathrm{d}\alpha \wedge \beta + \alpha \wedge \mathrm{d}\beta = \sum_{I,J} \mathrm{d}\alpha_I \beta_J e_I^* \wedge e_J^* + (-1)^k \sum_{I,J} \alpha_I \mathrm{d}\beta_J e_I^* \wedge e_J^*.$$

□

Corollary 5.86:

If $\alpha = \mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_k$, where $f_i \in C^\infty(U)$, then

$$\mathrm{d}\alpha = 0.$$

Definition 5.87: Pull-back

Given $\alpha \in \Omega^k(U)$, $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^m$ and a map $F : V \to U, C^\infty$; we define the PULL-BACK of α by F, denoted as $F^*(\alpha) \in \Omega^k(V)$ as follows:

$$(F^*\alpha)_x(A) = \alpha_{F(x)}(DF_x A).$$

Exercise 5.88:

Show that $F^*(\alpha)$ is indeed a k-form on V.

Lemma 5.89:

If $\alpha = e_I^*$, for some $I = (i_1, \ldots, i_k)$ and $F(y) = (F_1(y), \ldots, F_n(y))^T$, then $F^*(\alpha) = \mathrm{d}F_{i_1} \wedge \cdots \wedge \mathrm{d}F_{i_k}$.

Proof. We compute

$$\begin{aligned} F^*(\alpha)(A) &= \alpha_{F(x)}(DF_x A) = e_I^*(DF(x)A) \\ &= \det(E_I^T \cdot DF(x)A) = \det \begin{pmatrix} \partial_{i_1} F_1 & \ldots & \partial_{i_1} F_m \\ \vdots & & \vdots \\ \partial_{i_k} F_1 & \ldots & \partial_{i_k} F_m \end{pmatrix} \cdot A \\ &= \det \begin{pmatrix} \partial_{i_1} F_1 & \ldots & \partial_{i_1} F_m \\ \vdots & & \vdots \\ \partial_{i_k} F_1 & \ldots & \partial_{i_k} F_m \end{pmatrix} \cdot \det A \\ &= (\mathrm{d}F_{i_1} \wedge \cdots \wedge \mathrm{d}F_{i_k})(A). \end{aligned}$$

□

Proposition 5.90: Pullback commutes with $\wedge$ and d

Given $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^m$ open and $F : V \to U, C^\infty$, then:

1. $\forall \alpha \in \Omega^k(U)$, $\beta \in \Omega^l(U)$,

$$F^*(\alpha \wedge \beta) = F^*(\alpha) \wedge F^*(\beta).$$

2. $\forall \alpha \in \Omega^k(U)$,

$$F^*(\mathrm{d}\alpha) = \mathrm{d}F^*(\alpha).$$

Proof. It is enough to show it for $\alpha_x = f(x)e_I^*$ and $\beta_x = g(x)e_J^*$. Then, by definition of pull-back, we have

$$\begin{aligned} \alpha \wedge \beta &= f(x)g(x)e_I^* \wedge e_J^* \\ F^*(\alpha \wedge \beta)_y(T) &= f(F(y))g(F(y))(e_I^* \wedge e_J^*)(DF_y T) \end{aligned}$$

.

Similarly,

$$\begin{aligned} F^*\alpha \wedge F^*\beta &= f(F(y))e_I^* \wedge g(F(y))e_J^* \\ (F^*\alpha \wedge F^*\beta)_y(T) &= f(F(y))g(F(y))(e_I^* \wedge e_J^*)(DF_y T) \end{aligned}$$

.

For the second part, we compute

$$\begin{aligned} F^*(\mathrm{d}\alpha) &= F^*(\mathrm{d}f \wedge e_I^*) = F^*(\mathrm{d}f) \wedge F^*(e_I^*) \\ &= \mathrm{d}F^*f \wedge F^*(e_I^*) = \mathrm{d}F^*f \wedge e_I^* \\ &= \mathrm{d}F^*(fe_I^*) = \mathrm{d}F^*(\alpha). \end{aligned}$$

□

Definition 5.91:

Let $\Phi : V \to U$ be a k-dimensional parametrized submanifold with $M = \Phi(V)$. Let α be a k-form on U. We define the integral of α over M as

$$\int_M \alpha := \int_V \alpha_{\Phi(x)}(D\Phi_x)\mathrm{d}x.$$

Definition 5.92: Orientability

A k-dimensional submanifold $M \subset \mathbb{R}^n$ is ORIENTABLE if it is covered by local parametrizations with transition maps that have positive determinant.

In $\mathbb{R}^3$ a 2-dimensional submanifold M is orientable iff $\exists \nu : M \to \mathbb{R}^3$ such that $\nu(x)$ is a normal vector to M at x and is continuous. Lec 32

Proposition 5.93:

A $(n-1)$ dimensional submanifold $M \subset \mathbb{R}^n$ is orientable iff $\exists \nu : M \to \mathbb{S}^{n-1}$ such that $\nu(p) \perp T_pM$ and ν is continuous.

Proof. Given some local parametrization $\Phi : V \to M \cap U$, we can define $\nu_\Phi(\Phi(y))$ as the unique vector perpendicular to T_pM such that

$$\det(\nu_\Phi(\Phi(y)), \partial_1\Phi(y), \dots, \partial_{n-1}\Phi(y)) > 0.$$

We need to check that whenever $\Phi(y) = \Psi(\tilde{y})$, then

$$\nu_\Phi(\Phi(y)) = \nu_\Psi(\Psi(\tilde{y})).$$

But then, by chain rule, we have

$$(\nu_\Psi(\Psi(\tilde{y})), D\Psi_{\tilde{y}}) = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & D\Phi_y & \\ 0 & & & \end{pmatrix} (\nu_\Phi(\Phi(y)), D\Phi_y).$$

So it follows that $\nu_\Phi(\Phi(y)) = \nu_\Psi(\Psi(\tilde{y}))$. Hence, we can define $\nu(p) = \nu_\Phi(p)$ for any local parametrization Φ such that $p \in \Phi(V)$.

Thus, given a system of transition maps with positive determinant, we can define ν as above, and it is continuous.

Conversely, given ν continuous and $\tilde{\phi}_j$, maybe the transition maps are not > 0. For each j look at

$$\det(\nu(\tilde{\phi}_j(y)), D\tilde{\phi}_j(y)).$$

If this is > 0, do nothing. If this is < 0, then we can replace $\tilde{\phi}_j$ by $\tilde{\phi}_j \circ \rho$ where rho flips two coordinates. □

Definition 5.94: Integration of k-forms over k-dimensional submanifolds

Let M be compact, m-dim, and oriented and let $\Phi_i : V_i \to M \cap U_i$ be a system of parametrizations where U_i are Balles. Take η_i be the partition of unity such that $\operatorname{spt} \eta_i \subset U_i$. Then, we can define for $\alpha \in \Omega^m(U)$,

$$\int_M \alpha := \sum_i \int_{V_i} \alpha(D\Phi_i)\eta_i(\Phi(y))\mathrm{d}y.$$

Using pull-back, we can write

$$\int_M \alpha = \sum_i \int_{V_i} \Phi_i^*(\alpha\eta_i)(e_1, \dots, e_m)\mathrm{d}y.$$

Recall the divergence theorem: Given $U \subset \mathbb{R}^n$ a C^1 bounded domain,

$$\int_U \operatorname{div} F \mathrm{d}x = \int_{\partial U} F \cdot \nu \mathrm{d}\operatorname{vol}_{n-1}.$$

We want to transform vector field terminology to differential forms. For this, recall that the transpose of a vector field F is a 1-form. Now, how can we transform a vector field F into a $n-1$-form? For this, we can define

$$\alpha_x(T_1, \dots, T_{n-1}) = \det(F(x), T_1, \dots, T_{n-1}).$$

If we go from F to α, then we can write

$$\alpha_F = \sum_{i=1}^n (-1)^{i-1} F_i(x)\mathrm{d}x_1 \wedge \dots \wedge \widehat{\mathrm{d}x_i} \wedge \dots \wedge \mathrm{d}x_n.$$

Here the hat means that we omit the term. We can then see that

$$\mathrm{d}\alpha_F = \operatorname{div} F \cdot \mathrm{d}x_1 \wedge \dots \wedge \mathrm{d}x_n.$$

Theorem 5.95: Stokes Theorem

$\forall \alpha \in \Omega^{n-1}(\mathbb{R}^n)$, we have

$$\int_U \mathrm{d}\alpha = \int_{\partial U} \alpha,$$

where ∂U is oriented according to the exterior unit normal.

Proof. Exercise. □

Definition 5.96: Parametrized bounded submanifold with boundary

Given $U \subset V \subset \mathbb{R}^m$, and $\Phi : V \to \mathbb{R}^n$ a parametrized submanifold with $\Phi(V) = M$. We call $\Phi(U)$ a PARAMETRIZED BOUNDED SUBMANIFOLD WITH BOUNDARY.

The boundary $\partial M := \Phi(\partial U)$ is a parametrized submanifold of dimension $m-1$.

Notice, that $\partial_{\text{top}}\Phi(\overline{U}) = \Phi(\overline{U})$.

Theorem 5.97: Generalized Stokes Theorem

Given $M, \partial M$ as above, and $\alpha \in \Omega^{m-1}(\mathbb{R}^n)$, we have

$$\int_{\partial M} \alpha = \int_M \mathrm{d}\alpha,$$

where the orientation of ∂M is induced by the exterior unit normal to M.

Proof.

$$\int_{\partial M} \alpha = \int_{\partial U} \Phi^*(\alpha) = \int_U \mathrm{d}\Phi^*(\alpha) = \int_U \Phi^*(\mathrm{d}\alpha) = \int_M \mathrm{d}\alpha.$$

□

In the special setup of $n = 3$, $m = 2$, this is equivalent to

Corollary 5.98:

Given $\Sigma \subset \mathbb{R}^3$ a smoothly bounded 2 dimensional surface, and $F \in C^\infty(\mathbb{R}^3, \mathbb{R}^3)$ a vector field, then

$$\int_\Sigma \operatorname{rot} F \cdot \nu \mathrm{d}s = \int_{\partial\Sigma} F \cdot \mathrm{d}L,$$

where $\operatorname{rot} F = (\partial x, \partial y, \partial z) \wedge F$ is the curl of F.

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6 Ordinary Differential Equations

We will beginn with constant coefficient linear ODEs. The setup is given by a function $y = y(t)$ such that

$$y^{(m)} + a_{m-1}y^{(m-1)} + \cdots + a_1 y' + a_0 y = f \quad \forall t \in I. \tag{6.1}$$

The unknown is the function $y : I \to \mathbb{R}/\mathbb{C}$, where $I \subset \mathbb{R}$ is an interval. And given data $f : I \to \mathbb{R}$ is C^0. Furthermore, $a_0, \ldots, a_{m-1} \in \mathbb{R}/\mathbb{C}$ are constants. We call m the ORDER of the ODE.

Now, what does it mean that $y(t)$ solves the ODE?. $y \in C^m(I)$ and the equation holds for all $t \in I$. We want to see how we can find such a function y.

Definition 6.1: Characteristic polynomial

Given an ODE as in equation (6.1), we define the CHARACTERISTIC POLYNOMIAL as

$$p(x) = \sum_{k=0}^{m} a_k x^k.$$

Thus we can write our ODE compactly as

$$Ly = f, \quad L = p\left(\frac{d}{dt}\right).$$

Let us first find some solutions where $f = 0$.

Definition 6.2: Homogeneous ODE

The equation $Lz = 0$, where $z : I \to \mathbb{C}$ is called the HOMOGENEOUS ODE associated to L.

The solutions of the homogeneous ODE form a vector space, since if z, w solve $Lz = 0$ and $Lw = 0$, then for any $\alpha, \beta \in \mathbb{C}$, we have

$$L(\alpha z + \beta w) = \alpha L z + \beta L w = 0.$$

Define

$$V = \{z : I \to \mathbb{C}, C^m \mid Lz = 0\}.$$

To find solutions, try ANSATZ $z(t) = \exp(\alpha t)$ for some $\alpha \in \mathbb{C}$. Then, we compute

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^k z(t) = \alpha^k z(t) \Rightarrow Lz = p(\alpha)z(t).$$

If α is a root of p, then $z(t) = \exp(\alpha t)$ is a solution of the homogeneous ODE. So for a typical p with m distinct roots

$$\dim V \geq m,$$

because we have m linearly independent solutions.

What do we do when roots have multiplicity? Then, for $k \geq 2$,

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} - \alpha\right)^k (t^{k-1}e^{\alpha t}) = 0.$$

This can be shown by induction.

Lemma 6.3:

If α is a root of p with multiplicity m, then the functions

$$p(x) = \prod_{j=1}^{k} (x - \alpha_j)^{m_j}.$$

where $\alpha_1, \ldots, \alpha_k$ are the distinct roots of p, and m_j is the multiplicity of α_j, then the functions

$$p(\frac{\mathrm{d}}{\mathrm{d}t})(t^k \exp(\alpha_j t)) = 0, \forall k = 0, \ldots, m_j - 1.$$

Proof. We compute for fixed j, $0 \leq k < m_j$,

$$p(\frac{\mathrm{d}}{\mathrm{d}t}) = \underbrace{\prod_{l \neq j} \left(\frac{\mathrm{d}}{\mathrm{d}t} - \alpha_l\right)^{m_l}}_{:=Q_j(\frac{\mathrm{d}}{\mathrm{d}t})} \cdot \left(\frac{\mathrm{d}}{\mathrm{d}t} - \alpha_j\right)^{m_j}.$$

We can write

$$p(\frac{\mathrm{d}}{\mathrm{d}t})(t^k e^{\alpha_j t}) = Q_j(\frac{\mathrm{d}}{\mathrm{d}t}) \cdot \left(\frac{\mathrm{d}}{\mathrm{d}t} - \alpha_j\right)^{m_j} (t^k e^{\alpha_j t}).$$

Since the second part is 0, we get $p(\frac{\mathrm{d}}{\mathrm{d}t})(t^k e^{\alpha_j t}) = 0$.

We conclude, that in any case, $\dim V \geq \deg p = m$. □

Proposition 6.4:

Let $L = p(\frac{\mathrm{d}}{\mathrm{d}t})$ with p as above and let $t_0 \in I$. Consider the INITIAL VALUE PROBLEM

$$\begin{cases} Lz(t) = 0 & \forall t \in I \\ z^{(k)}(t_0) = w_k & \forall k = 0, \ldots, m-1 \end{cases}.$$

Then, given $w_i \in \mathbb{C}$, there is a unique solution.

Lemma 6.5: Zero-In, Zero-Out

In the setup of the proposition, $w_i = 0 \ \forall i$ and $z : I \to \mathbb{C}$ solves the initial value problem, then $z = 0$.

Lemma 6.6: Gronwall's inequality

If $u \in C^1([a, b])$ satisfies

$$u'(t) \leq \beta(t)u(t) \forall t \in [a, b], \text{ where } \beta \in C^0([a, b]).$$

Then

$$u(t) \leq u(a) \exp\left(\int_a^t \beta(s) \mathrm{d}s\right) \forall t \in [a, b].$$

Proof. Define $v(t) := \exp\left(\int_a^t \beta(s)\mathrm{d}s\right) \in C^1$. Notice, that $v > 0$ and

$$v'(t) = \beta(t)v(t).$$

Take $\frac{u}{v} = \phi(t)$, so ϕ has derivative

$$\phi'(t) = \frac{u'(t)v(t) - u(t)v'(t)}{v(t)^2} \leq \frac{\beta(t)u(t)v(t) - u(t)\beta(t)v(t)}{v(t)^2} = 0.$$

So ϕ is non-increasing, and thus $\phi(t) \leq \phi(a) \forall t \in [a, b]$. Hence, by mean value theorem, we have

$$\phi(t) - \phi(a) = \phi'(\xi)(t - a) \leq 0.$$

Plugging in the definitions of ϕ and v, we get

$$u(t) \leq u(a) \exp\left(\int_a^t \beta(s) \mathrm{d}s\right) \forall t \in [a, b].$$

□

Proof. [Lemma 6.5] Consider $u(t) \coloneqq \sum_{i=0}^{m-1}(w^{(i)}(t))^2$. Then,

$$u'(t) = \sum_{i=0}^{m-2} 2w^{(i)}(t)w^{(i+1)}(t) + 2w^{(m-1)}(t)w^{(m)}(t).$$

So $w^{(m)}(t) = -\sum_{i=0}^{m-1} a_i w^{(i)}(t)$, and thus

$$|w^{(m)}(t)| \leq C \cdot \sum_{i=0}^{m-1} |w^{(i)}(t)|.$$

Using the inequality $2ab \leq a^2 + b^2$, we get

$$\begin{aligned} u'(t) &\leq \sum_{i=0}^{m-2} w^{(i)}(t)^2 + \sum_{i=0}^{m-2} w^{(i+1)}(t)^2 + 2C\left(\sum_{i=0}^{m-1} w^{(i)}(t)\right)^2 \\ &\leq 2u(t) + 2Cm \cdot u(t) = \underbrace{2(1+cm)}_{:=\bar{c}} u(t). \end{aligned}$$

Applying Gronwall's inequality (Lemma 6.6), we get

$$0 \leq u(t) \leq u(t_0)\exp(\bar{c}(t-t_0)) = 0 \forall t \in I.$$

□

Proof. [Proposition 6.4] Let $T : V \to \mathbb{R}^m$ be the map

$$z(t) \mapsto (z(t_0), z'(t_0), \ldots, z^{(m-1)}(t_0)).$$

This map is a homomorphism. By lemma 6.5, $\ker T = \{0\}$, so by rank-nullity, $\dim V \leq m$. Since we already know $\dim V \geq m$, we get $\dim V = m$ and T is an isomorphism. Hence, given $w_i \in \mathbb{C}$, there is a unique solution to the initial value problem. □

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We will quickly show that our polynomial exponential functions are indeed linearly independent.

Proof. The set $\{p_j(t)e^{\alpha_j(t)}\}$ is linearly independent. Indeed assume by contradiction,

$$\sum_{j=1}^{N} q_j(t)e^{\alpha_j t} = 0 \quad \forall t \in I.$$

Choose a linear combination as this with N minimal.

We multiply our polynomial by $xe^{-\alpha_N t}$, and we get

$$\sum_{j=1}^{N-1} q_j(t)e^{(\alpha_j - \alpha_N)t} + q_N(t) = 0 \quad \forall t \in I.$$

Differentiating $\deg q_N + 1$ times, we get

$$\sum_{j=1}^{N-1} \tilde{q}_j(t)e^{(\alpha_j - \alpha_N)t} = 0 \quad \forall t \in I.$$

If we now divide by $e^{\alpha_N t}$, we get

$$\sum_{j=1}^{N-1} \tilde{q}_j(t)e^{\alpha_j t} = 0 \quad \forall t \in I.$$

Which contradicts the minimality of N. □

We now want to solve the inhomogeneous equation

$$Ly(x) = f(x). \tag{6.2}$$

If y_1, y_2 are solutions of (6.2), then

$$L(y_1 - y_2) = Ly_1 - Ly_2 = f - f = 0.$$

Thus the set of functions solving the inhomogeneous equation is the affine space

$$\{y_{\text{part}} + z | z \in V\},$$

where y_{part} is any particular solution of (6.2). Now how do we find a particular solution?

If $f(t) = q(t)e^{\omega t}$, with $\deg(q) = m$ there is the following algorithm.

If ω is not a root of p, we can try the Ansatz

$$y(t) = Q(t)e^{\omega t} \text{ with } \deg(Q) = m.$$

If ω is a root of p with multiplicity k, we can try the Ansatz

$$y(t) = Q(t)t^k e^{\omega t} \text{ with } \deg(Q) = m.$$

If $f(t)$ is a general poly-exp function, apply the superposition principle.

If y_1, y_2 solve $Ly_i = f_i$ with $y_i^{(k)} = w_{i,k}$, then $a_1y_1 + a_2y_2$ with $a_1, a_2 \in \mathbb{C}$ solves $L(y_1 + y_2) = f_1 + f_2$ with initial conditions

$$(a_1y_1 + a_2y_2)^{(k)} = a_1w_{1,k} + a_2w_{2,k}.$$

Example 6.7: Pumped harmonic oscillator

Consider the ODE

$$\begin{cases} y'' + ky = \sin(\omega t) \\ y(0) = 0, y'(0) = 0 \end{cases}.$$

Where $k > 0$ and $\omega \in \mathbb{R}$.

Solution. The corresponding homogeneous ODE is $y'' + ky = 0$, which has characteristic polynomial $p(x) = x^2 + k$. This polynomial has complex roots

$$\alpha_1 = i\sqrt{k}, \quad \alpha_2 = -i\sqrt{k}.$$

So we find our fundamental system of solutions to the homogeneous ODE as

$$V = \text{Sp}\{\exp(\sqrt{k}it), \exp(-\sqrt{k}it)\}.$$

By definition $\sin(t) = \frac{\exp(it)-\exp(-it)}{2i}, \cos(t) = \frac{\exp(it)+\exp(-it)}{2}$. So we can write

$$V = \text{Sp}\{\cos(\sqrt{k}t), \sin(\sqrt{k}t)\}.$$

So we get the solution to the homogeneous ODE as

$$y(t) = A\sin(\sqrt{k} \cdot t) + B\cos(\sqrt{k} \cdot t) + y_{\text{part}}(t).$$

Notice, that $f(t) = \sin(\omega t) = \frac{\exp(i\omega t)-\exp(-i\omega t)}{2i}$ is a poly-exp function. In the approach above, we have $\tilde{\omega} = \pm i\omega$. We have seen that the roots of p are $\pm i\sqrt{k}$. So if $\omega \neq \sqrt{k}$, we can try the Ansatz

$$y_{\text{part}}(t) = \tilde{C}e^{i\omega t} + \tilde{D}e^{-i\omega t} = C\sin(\omega t) + D\cos(\omega t).$$

We now compute Ly_{part} where $L = \frac{\mathrm{d}^2}{\mathrm{d}t} + k$ and we get

$$Ly_{\text{part}} = (k - \omega^2)(C\sin(\omega t) + D\cos(\omega t)) \stackrel{!}{=} \sin(\omega t).$$

Solving this in the basis $\{\sin(\omega t), \cos(\omega t)\}$, we get

$$C = \frac{1}{k - \omega^2}, D = 0.$$

So our solution is

$$y(t) = A\sin(\sqrt{k} \cdot t) + B\cos(\sqrt{k} \cdot t) + \frac{\sin(\omega t)}{k - \omega^2}.$$

Imposing the initial conditions, we get $B = 0$ and $A = \frac{-\omega}{\sqrt{k}(k-\omega^2)}$. Hence,

$$y(t) = \frac{-\omega}{\sqrt{k}(k - \omega^2)}\sin(\sqrt{k} \cdot t) + \frac{\sin(\omega t)}{k - \omega^2}.$$

Exercise 6.8:

Compute the limit, when $\omega \to \sqrt{k}$, of the solution $y(t)$ for fixed t.

Definition 6.9: Pulse at 0

$$\begin{cases} Lz = 0 \\ z^{(k)}(0) = 0 \forall k = 0, \dots, m-2 \\ z^{(m-1)}(0) = 1 \end{cases}.$$

We denote the unique solution $z(t)$ to this IVP $P(t)$ and call it PULSE AT 0 for L.

6.1 Duhemel Principle

Suppose we want to solve $Ly(t) = f(t)$, where $f : I \to \mathbb{C}$ continuous given, with initial conditions $y^{(k)}(0) = 0$ for $k = 0, \dots, m-1$. Then, the solution is given by

$$y(t) = \int_{t_0}^{t} f(s)P(t-s)\mathrm{d}s.$$

For example, if $f(t) = \delta_S(t)$, then by Zero-In, Zero-Out, we have $y(t) = 0$ before the pulse, and $y(t) = P(t)$ after the pulse.

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The solution to the exercise from the last lecture is given by

$$y(t) = \frac{1}{2\sqrt{k}}\cos(\sqrt{k}t)t + \frac{1}{2k}\sin(\sqrt{k}t).$$

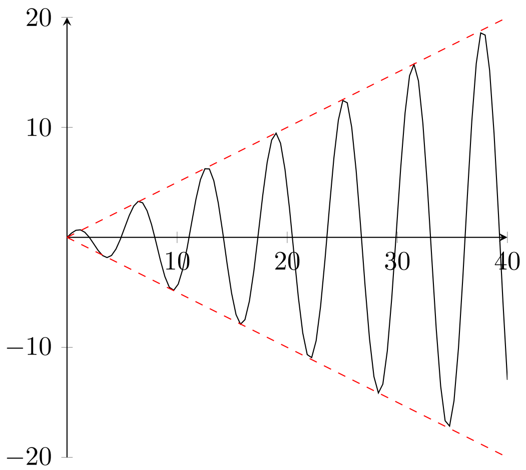

Figure 22

Theorem 6.10: Duhamels' Principle

Given the ODE

$$\begin{cases} Ly(t) = f(t) \\ y(t_0) = y'(t_0) = \cdots = y^{(m-1)}(t_0) = 0 \end{cases}.$$

Then, the solution is given by

$$y(t) = \int_{t_0}^{t} f(s)P(t-s)\mathrm{d}s.$$

Where P is a pulse at 0.

Proof. Let y as in the theorem. We want to compute y'. In general, given $F(t_1, t_2) := \int_0^{t_1} h(s, t_2)\mathrm{d}s$, we have assuming $h, \partial_{t_2} h$ are continuous, then

$$\partial_{t_1} F(t_1, t_2) = h(t_1, t_2), \quad \partial_{t_2} F(t_1, t_2) = \int_0^{t_1} \partial_{t_2} h(s, t_2)\mathrm{d}s.$$

Hence, by chain rule,

$$y'(t) = \int_{t_0}^{t} f(s)P'(t-s)\mathrm{d}s.$$

Applying the same argument, we get

$$y''(t) = \int_{t_0}^{t} f(s)P''(t-s)\mathrm{d}s.$$

Inductively, we get

$$y^{(m)}(t) = f(t)p^{(m-1)}(t-t) + \int_{t_0}^{t} f(s)P^{(m)}(t-s)\mathrm{d}s.$$

Since P is a pulse at 0, we have $P^{(m-1)}(0) = 1$ and thus, $y^{(m)}(t) = f(t)$.

Summing our identities, we get

$$\begin{aligned} Ly(t) &= \sum_{i=0}^{m} a_i y^{(i)}(t) \\ &= \sum_{i=0}^{m} a_i \int_{t_0}^{t} f(s)P^{(i)}(t-s)\mathrm{d}s + f(t) \\ &= \int_{t_0}^{t} f(s)\sum_{i=0}^{m} a_i P^{(i)}(t-s)\mathrm{d}s + f(t) \\ &= \int_{t_0}^{t} f(s)LP(t-s)\mathrm{d}s + f(t) = f(t). \end{aligned}$$

□

6.2 Systems of ODEs

Given $I \subset \mathbb{R}$ an interval with nonempty interior and $U \subset I \times \mathbb{R}^n$ open, we are given $F : U \to \mathbb{R}^n$, continuous. We are interested in the equation

$$u'(t) = F(t, u(t)). \tag{6.3}$$

We will solve it with some initial conditions $u(t_0) = x_0$. Solving this equation means finding a interval $\tilde{I} \subset I$ with and $u : \tilde{I} \to \mathbb{R}^n$ such that $u(t) \in U$ is differentiable and satisfies (6.3).

For ODEs of order $m > 1$ the corresponding system of ODEs is given by

$$y^{(m)}(t) = F(t, y^{(m-1)}(t), \dots, y(t)).$$

This equation can be transformed into 1st order system of ODEs by setting

$$\begin{aligned} u : \mathbb{R} &\longrightarrow (\mathbb{R}^n)^m \\ t &\longmapsto u(t) = \begin{pmatrix} y(t) \\ y'(t) \\ \vdots \\ y^{(m-1)}(t) \end{pmatrix}. \end{aligned}$$

Then, we can write

$$u'(t) = \tilde{F}(t, u(t)).$$

Furthermore we enforce, that $y^{(i)'} = y^{(i+1)}$ for $i = 0, \dots, m-2$, and $y^{(m-1)'} = F(t, y^{(m-1)}, \dots, y)$.

As such, we only focus on solving 1st order systems of ODEs. If we understand this case, we can understand the general case. We begin with the simplest case of (6.3) that corresponds to that of constant coefficients. I.e.

$$F(t, u(t)) = Au(t) \quad A \in M_{n \times n}(\mathbb{R}).$$

Notice, if $n = 1$, then A is just a number, and the solution is given by $u(t) = e^{A(t-t_0)}u(t_0)$.

To replicate this solution for $n > 1$, we need to define the MATRIX EXPONENTIAL.

Definition 6.11: Matrix exponential

Given $A \in M_{n\times n}(\mathbb{R})$, we define

$$\exp(A) = \sum_{k=0}^{\infty} \frac{A^k}{k!}.$$

This definition is well-defined since

$$\left\|\sum_{i=0}^{N} \frac{A^j}{j!}\right\|_2 \le \sum_{j=0}^{N} \frac{\|A\|_2^j}{j!} \le e^{\|A\|_2}.$$

Observe, that if A, B commute, then $\exp(A + B) = \exp(A)\exp(B)$. Furthermore, if S is invertible, then $\exp(SAS^{-1}) = S\exp(A)S^{-1}$. In particular, if D is diagonal, then $\exp(D)$ is the diagonal matrix with entries $\exp(D_{ii})$. As such given a matrix A we can compute the Jordan normal form of A, and then compute the exponential of the Jordan normal form, and then conjugate back to get $\exp(A)$.

Proposition 6.12:

The unique solution of

$$\begin{cases} u'(t) = Au(t) \\ u(t_0) = x_0 \end{cases},$$

is given by

$$u(t) = \exp(A(t-t_0))x_0.$$

Proof. Define $v(t) = \exp(At)$. Then

$$v'(0) = \lim_{t\to 0} \frac{\exp(At) - I}{t} = A.$$

Thus,

$$u(t+h) = v(h)u(t) = \exp(A(t-t_0))\exp(Ah).$$

From this it follows, that

$$u'(t) = \lim_{h\to 0} \frac{u(t+h) - u(t)}{h} = Au(t).$$

This shows that u is indeed a solution. To show uniqueness, we can consider $z(t) = \exp(-A(t-t_0))u(t)$. Computing z', we get $z' = 0$.
Lec 36 Hence, $z(t) = z(t_0) = x_0$. Thus, $u(t) = \exp(A(t-t_0))x_0$. □

Example 6.13:

Consider the system of ODEs

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

So

$$y' = F(t,y) = F(y) = \begin{pmatrix} -y_2 \\ y_1 \end{pmatrix}.$$

We find the characteristic polynomial of A is given by $\lambda^2 + 1$. Hence, the eigenvalues are $\pm i$. The Jordan normal form of A is given by

$$J = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

So, we can compute $\exp(Jt)$ as

$$e^{Jt} = \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} = \begin{pmatrix} \cos t + i\sin t & 0 \\ 0 & \cos t - i\sin t \end{pmatrix}$$

Hence, we can compute $\exp(At)$ as

$$\exp(At) = Se^{Jt}S^{-1} = \begin{pmatrix} -\cos t & +\sin t \\ -\sin t & -\cos t \end{pmatrix}.$$

Example 6.14:

Solve the system of ODEs $y' = By$, where

$$B = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

Notice, that $B = A + I$, where A is the matrix from the previous example. Hence, we can compute $\exp(Bt) = e^t \exp(At)$.

We now want to solve a general ODE of the form $y' = F(t,y)$. The trick for this is to rewrite the ODE in an integral form assuming $y \in C^1$

$$y(t) = \int_{t_0}^{t} F(s, y(s))\mathrm{d}s + x_0. \tag{6.4}$$

Notice, that this integral is meant component-wise. So we have $y(t) = Ty(t)$, where T is a function such that

$$Ty(t) = \int_{t_0}^{t} F(s, y(s))\mathrm{d}s + x_0. \tag{6.5}$$

The idea is, that this map will be $T : X \to X$ a contraction from a suitable space X to itself, and then we can apply Banach's fixed point theorem to get a unique fixed point of T, which will be the solution to our ODE.

Definition 6.15:

If (Y, d) is a metric space, we define

$$C_b(Y, \mathbb{R}^n) := \{f : Y \to \mathbb{R}^n : f \text{ cont. and bdd.}\}.$$

This space is a normed vector space with the norm

$$\|f\|_\infty = \sup_{y\in Y} |f(y)|.$$

Definition 6.16: Uniform Convergence

Given $f, f_k : Y \to \mathbb{R}^n$ continuous we say that

$$f_k \xrightarrow{unif.} f,$$

if $\|f_k - f\|_\infty \to 0$ as $k \to \infty$.

Proposition 6.17:

$C_b(Y, \mathbb{R}^n)$ is a complete metric space.

Lemma 6.18:

The uniform limit of continuous functions is continuous.

Proof. Assume $f_k \xrightarrow{unif.} f$. Let $\varepsilon > 0$. Then, there exists N such that for all $k \geq N$, $\|f_k - f\|_\infty < \varepsilon/3$. In particular, for all $y \in Y$, we have $|f_k(y) - f(y)| < \varepsilon/3$. Since f_N is continuous, there exists $\delta > 0$ such that if $d(y, y') < \delta$, then $|f_N(y) - f_N(y')| < \varepsilon/3$. Hence, if $d(y, y') < \delta$, then

$$|f(y)-f(y')| \leq |f(y)-f_N(y)|+|f_N(y)-f_N(y')|+|f_N(y')-f(y')| < \varepsilon.$$

□

Lemma 6.19:

A cauchy sequence in $C_b(Y, \mathbb{R}^n)$ converges uniformly.

Proof. Let f_k be a cauchy sequence in $C_b(Y, \mathbb{R}^n)$. Then, for all $\varepsilon > 0$, there exists N such that for all $k, l \geq N$, $\|f_k - f_l\|_\infty < \varepsilon$.

In particular, for all $y \in Y$, we have $|f_k(y) - f_l(y)| < \varepsilon$. Hence, for all $y \in Y$, the sequence $f_k(y)$ is a cauchy sequence in $\mathbb{R}^n$, and thus converges to some point $f(y) \in \mathbb{R}^n$.

f is bounded since we can apply $\|f_k - f_l\|_\infty < \varepsilon$ for $\varepsilon = 1$. So $|f_k(y)| \leq 1 + |f_N(y)|$ for all $k \geq N$. Letting $k \to \infty$, we get $|f(y)| \leq 1 + |f_N(y)|$ for all $y \in Y$.

Recall, that $f_k \to f$ if $\|f_k - f\|_\infty \to 0$. We have seen that $\|f_k - f_l\|_\infty < \varepsilon$ for all $k, l \geq N$. Hence, sending k to infinity, we get $\|f_l - f\|_\infty \leq \varepsilon$ for all $l \geq N$. Hence, $\|f_k - f\|_\infty \to 0$ as $k \to \infty$. □

Proof. [of Proposition 6.17] By Lemma 6.19, every Cauchy sequence converges uniformly to a bounded function, which by lemma 6.18 is continuous. Hence, $C_b(Y, \mathbb{R}^n)$ is complete. □

So now we can define the space X as $C_b([t_0 - \delta, t_0 + \delta], \mathbb{R}^n)$ for some $\delta > 0$. We define $\tilde{x}_0(t) = x_0$ for all t and thus $X = B_r(\tilde{x}_0)$. In other words

$$X = \{y = y(t) \mid \|y - \tilde{x}_0\|_\infty < r\}.$$

If $y \in X$, we define

$$C_b([t_0 - \delta, t_0 + \delta], \mathbb{R}^n) \ni Ty(t) = \int_{t_0}^t F(s, y(s)) \mathrm{d}s + x_0.$$

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Proposition 6.20: Local Existance of solutions

In the setup above, assume $\mathcal{C} = [t_0 - r, t_0 + r] \times B_r(x_0) \subset U$ for some $r > 0$, and F is bounded by C and Lipshitz in the second variable, i.e.

$$|F(t, y) - F(t, y')| \leq L|y - y'| \forall (t, y), (t, y') \in \mathcal{C}.$$

Then, putting $I = (t_0 - \delta, t_0 + \delta)$, where $0 < \delta \leq \min\{r, \frac{r}{2C}, \frac{1}{2L}\}$, the equation equations (6.3) and (6.4) are equivalent, and there exists a unique solution $u : I \to B_r(x_0)$.

Proof. Let T be defined as before. Let us specify the domain.

$$V := \{v : I \to \overline{B_r(x_0)} \in C^0 \mid |v(t) - x_0| \leq r \forall t \in I\}.$$

Note, that $T : V \to C_b(I, \mathbb{R}^n)$, we claim $T(V) \subset V$. Indeed,

$$\begin{aligned} |(Tv)(t) - x_0| &= \left| \int_{t_0}^t F(s, v(s)) \mathrm{d}s \right| \\ &\leq |t - t_0| \cdot C \qquad\qquad t \in I \\ &\leq \delta C \leq \frac{r}{2} < r. \end{aligned}$$

so indeed $T : V \to V$. Furthermore V is a closed subset of the complete space $C_b(I, \mathbb{R}^n)$, and thus is complete.

Let us now show T is a contraction. Let $u, v \in V$. Then,

$$\begin{aligned} \|Tu_1(t) - Tu_2(t)\| &= \left| \int_{t_0}^F (s, u_1(s)) - F(s, u_2(s)) \mathrm{d}s \right| \\ &\leq \int_{t_0}^t |F(s, u_1(s)) - F(s, u_2(s))| \,\mathrm{d}s \\ &\leq L \int_{t_0}^t |u_1(s) - u_2(s)| \mathrm{d}s \\ &\leq \delta L \|u_1 - u_2\|_\infty \leq \frac{1}{2} \|u_1 - u_2\|_\infty. \end{aligned}$$

Since this holds for all $t \in I$, we get $\|Tu_1 - Tu_2\|_\infty \leq \frac{1}{2}\|u_1 - u_2\|_\infty$. Hence, T is a contraction.

Applying Banach's fixed point theorem, we get that T has a unique fixed point $u \in V$. This shows that there exists a unique C^0 solution to (6.4). But then, if $u : I \to B_r(x_0)$ were some C^1 solution of (6.3), then u would also be a solution to (6.4). Hence, u is the unique solution to (6.3). □

Example 6.21:

We want to show why the Lipshitz condition is necessary.

Consider the ODE

$$\begin{cases} u(t) = t^\gamma \\ u'(t) = \gamma t^{\gamma - 1} = \gamma (t^\gamma)^{\frac{\gamma - 1}{\gamma}} \end{cases}.$$

Where $\gamma > 1$.

Solution. We can write

$$u'(t) = F(u(t)) \quad F(s) = \gamma (s)^{\frac{\gamma - 1}{\gamma}}.$$

Let us enforce $u(0) = 0$. Then we have two solutions to this ODE, $u(t) = 0$ and $u(t) = t^\gamma$.

Definition 6.22: Maximal Interval

An interval I is a MAXIMAL INTERVAL for the solution u of an ODE if u is a solution on I, and if u is a solution on some interval I', then $I' \subset I$.

Theorem 6.23: Cauchy-Lipshitz

Given $U \subset \mathbb{R} \times \mathbb{R}^n$ open, $F : U \to \mathbb{R}^n$ C^0 in t and C^1 in x $(F = F(t, x))$. Let $(t_0, x_0) \in U$. Then, there exists a maximal interval $I = (a, b) \ni t_0$ and a unique $u : I \to \mathbb{R}^n$ such that $(t, u(t)) \subset U \forall t \in I$ and $u \in C^1$ satisfies

$$\begin{cases} u'(t) = F(t, u(t)) \quad \forall t \in I \\ u(t_0) = x_0 \end{cases}.$$

Consider the set of all pairs I, u_I where I is an open interval containing t_0 and $u_I : I \to \mathbb{R}^n$ satisfies $(t, u_I(t)) \in U$ for all $t \in I$ and u_I is a C^1 solution to the ODE on I. The key observation is that if (I, u_I) and (J, u_J) are two pairs in this set, then u_I and u_J coincide on $I \cap J$.

Proof. Let $I \cap J$ be an open, connected interval. Let $A \subset I \cap J$ be the set of points, where u_I and u_J coincide. Then $I \cap J \setminus A$ is open, and thus A is closed.

Furthermore, A is open. Indeed, let $t_1 \in A$, $x_1 = u_I(t_1) = u_J(t_1)$. Since $(t_1, x_1) \in U$, there exists a cylinder $\mathcal{C} = t_1 - \delta, t_1 + \delta \times \overline{B_\varepsilon(x_1)} \subset U$. Applying the local existence and uniqueness theorem (Prop. 6.20), $\exists \delta > 0$ such that there is a unique solution u to the ODE on $(t_1 - \delta, t_1 + \delta)$ with $u(t_1) = x_1$. Since u_I and u_J are both solutions to the ODE on $(t_1 - \delta, t_1 + \delta)$ with the same initial condition, we get that u_I and u_J coincide on $(t_1 - \delta, t_1 + \delta)$.

Hence, A is open. Since A is nonempty, open and closed in the connected set $I \cap J$, we get $A = I \cap J$. Hence, u_I and u_J coincide on $I \cap J$. □

We define $I_max = \bigcup I$ and $u(t) \coloneqq u_I(t)$. By construction, u is a solution to the ODE on I_max. To show that u is unique everywhere, we can just apply the same argument as before to show that if u' is another solution on some interval I', then u and u' coincide on $I' \cap I_max$. Hence, u is the unique solution on I_max.

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