# Analysis I

**Def (Field):** Commutative Ring with inverse under $\cdot$ $\forall x \neq 0$.

**Def (Order Relation):** Given set $X$, $R \subseteq X \times X$, such that $x \leq x \ \forall x$, $x \leq y \wedge y \leq z \Rightarrow x \leq z$, $x \leq y \wedge y \leq x \Rightarrow x = y$.

**Def (Ordered Field):** $(F, \leq)$ such that $x \leq y \vee y \leq x$, $x \leq y \Rightarrow x + z \leq y + z$, $0 \leq x \wedge 0 \leq y \Rightarrow 0 \leq xy$.

**Lemma:** Given $(F, \leq)$, $\mathbb{Z} \subsetneq \mathbb{Q} \subseteq F$.

**Lemma (Triangle Inequality):** $(F, \leq)$. $|x+y| \leq |x| + |y|$.

**Lemma (Inverse Triangle Inequality):** $(F, \leq)$. $||x| - |y|| \leq |x - y|$.

**Def (Completeness Axiom):** $(F, \leq)$. Let $X, Y \subseteq F$ non-empty s.t. $x \leq y$. Then $\exists c \in F: x \leq c \leq y \ \forall x, y$.

**Def (Real numbers):** Any complete, ordered field.

**Def (Open set):** $U \subseteq \mathbb{R}$ is open if $\forall x \in U \ \exists \varepsilon > 0: (x - \varepsilon, x + \varepsilon) \subseteq U$.

**Def (Closed set):** $V \subseteq \mathbb{R}$ is closed if $\mathbb{R} \setminus V$ is open.

**Def (Supremum):** Let $X \subseteq \mathbb{R}$. $\sup(X) = \min\{a \in \mathbb{R} \mid a \geq x \ \forall x \in X\}$.

**Lemma:** If $X$ is bounded, $\sup(X)$ exists.

**Thm (Archimedian Principle):** $x \in \mathbb{R} \ \exists! n \in \mathbb{Z}: n \leq x < n+1$.

**Corollary:** $\forall \varepsilon > 0 \ \exists n \in \mathbb{N}: \frac{1}{n} < \varepsilon$.

**Def (Sequence):** $f: \mathbb{N} \to \mathbb{R}$.

**Def (Limit of seq):** $(x_n)_{n \geq 0}$ CONVERGES if $\exists A: \forall \varepsilon > 0 \ \exists N: |x_n - A| < \varepsilon \ \forall n \geq N$. We write $\lim_{n \to \infty} x_n = A$.

**Def (Accumulation Point):** $A \in \mathbb{R}: \forall \varepsilon > 0 \ \forall N \in \mathbb{N} \ \exists n \geq N: |x_n - A| < \varepsilon$.

**Thm (Bolzano-Weierstrass):** $A \in \mathbb{R}$ acc. pt. $\Leftrightarrow \exists (x_{n_k})_{k \geq 0}$ s.t. $\lim_{k \to \infty} x_{n_k} = A$.

**Thm (Limit Theorem):** $(x_n)_{n \geq 0} \to A$, $(y_n)_{n \geq 0} \to B$. Then $x_n + y_n \to A + B$, $x_n \cdot y_n \to A \cdot B$, $\alpha \cdot x_n \to \alpha \cdot A$, $x_n^{-1} \to A^{-1}$.

**Thm (Limit Inequalities):** $A < B \Rightarrow \exists N: x_n < y_n \ \forall n \geq N$; $x_n \leq y_n \Rightarrow A \leq B$.

**Lemma:** Given $x_n \leq y_n \leq z_n$ and $x_n, z_n \to A \Rightarrow y_n \Rightarrow A$.

**Lemma:** A monotone seq converges $\Leftrightarrow$ it is bounded.

**Def (Cauchy seq):** $\forall \varepsilon > 0 \ \exists N: |x_n - x_m| < \varepsilon \ \forall n, m \geq N$.

**Def (Continuity):** $\forall x_0 \ \forall \varepsilon > 0 \ \exists \delta > 0: |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$.

**Lemma:** If $f_1, f_2$ are continuous, so is $f_1 + f_2$, $f_1 \cdot f_2$, $\alpha \cdot f_1$.

**Lemma:** Polynomials are continuous.

**Lemma:** If $f, g$ are continuous so is $g \circ f$.

**Thm (Sequential continuity):** $f$ is continuous $\Leftrightarrow \forall (x_n)_{n \geq 0} \subseteq D$, $x_n \to \bar{x} \Rightarrow f(x_n) \to f(\bar{x})$.

**Thm (Intermediate Value Theorem):** $f: [a,b] \to \mathbb{R}$ continuous. If $f(a) < f(b)$ Then $\forall c \in [f(a), f(b)] \ \exists \bar{x} \in [a,b]: f(\bar{x}) = c$.

**Thm (Inverse function theorem):** $f: I \to \mathbb{R}$ continuous, strictly monotone. Then $f(I)$ is an interval and $f: I \to f(I)$ has a continuous, strictly monotone inverse $f^{-1}$.

**Lemma:** Every bounded sequence has a convergent subsequence.

**Theorem:** $f: [a,b] \to \mathbb{R}$ continuous takes maximum and minimum.

**Def (Uniform continuity):** $\forall \varepsilon > 0 \ \exists \delta > 0: \forall x, y \ |x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$.

**Theorem:** $f: [a,b] \to \mathbb{R}$ continuous $\Rightarrow$ uniform continuity.

**Def (Exponential):** $\exp(x) = \lim_{n \to \infty} (1 + \frac{x}{n})^n$.

**Def (Limit):** $f: D \to \mathbb{R}$, $x_0$ acc. pt of $D$. $\lim_{x \to x_0} f(x) = L \in \mathbb{R}$ if $\forall \varepsilon > 0 \ \exists \delta > 0: x \in D$, $|x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon$.

**Rem:** $\lim_{x \to x_0} f(x) = L$, $\lim_{x \to x_0} g(x) = M \Rightarrow f + g \to L + M$; $f \cdot g \to L \cdot M$; $f \leq g \to L \leq M$.

**Rem:** Continuity at $x_0 \Leftrightarrow \lim_{x \to x_0} f(x) = f(x_0)$.

**Prop:** $f: D \to E$, $L = \lim_{x \to \bar{x}} f(x) = L \in E$. $g: E \to \mathbb{R}$ cont at $L \Rightarrow \lim_{x \to \bar{x}} g(f(x)) = g(L)$

**Def (Big O):** $f, g: D \to \mathbb{R}$. $f(x) = O(g(x))$ as $x \to x_0$ if $|\frac{f}{g}| \leq M$ for $|x - x_0| < \delta$

**Def (Little o):** $f, g: D \to \mathbb{R}$. $f(x) = o(g(x))$ as $x \to x_0$ if $|\frac{f}{g}| \to 0$ as $x \to x_0$

**Def (Pointwise convergence):** $(f_n)_{n \geq 0} \to f$ pointwise if $\forall x \in D$, $f_n(x) \xrightarrow{n \to \infty} f(x)$. Alternatively $\forall x \in D \ \forall \varepsilon > 0 \ \exists N \in \mathbb{N}: |f_n(x) - f(x)| < \varepsilon \ \forall n \geq N$.

**Def (Uniform convergence):** $f_n \to f$ uniformly if $\forall \varepsilon > 0 \ \exists N \in \mathbb{N}: |f_n(x) - f(x)| < \varepsilon \ \forall x \in D, n \geq N$

**Thm:** If $f_n \to f$ uniformly and all $f_n$ (uniformly) cont $\Rightarrow f$ (uniformly) continuous.

**Prop:** If $\sum_{n=0}^{\infty} a_n$ converges, $\lim_{n \to \infty} a_n = 0$.

**Lemma:** $\sum_{n=0}^{\infty} q^n$ converges iff $|q| < 1$.

**Lemma:** $\sum_{n=0}^{\infty} a_n$ converges iff $\sum_{n=N}^{\infty} a_n$ converges for some $N \in \mathbb{N}$

**Prop:** Let $0 \leq a_k \leq b_k \ \forall k \in \mathbb{N}$. Then $0 \leq \sum_{n=0}^{\infty} a_n \leq \sum_{n=0}^{\infty} b_n$.

**Prop (Cauchy Condensation):** Let $a_k \geq 0$, decreasing. Then $\sum_{k=0}^{\infty} a_k$ converges $\Leftrightarrow \sum_{k=0}^{\infty} 2^k a_k$ converges

**Def (Absolute Convergence):** $\sum a_k$ converges absolutely if $\sum |a_k|$ converges.

**Def (Conditional Convergence):** $\sum a_k$ converges but $\sum |a_k| = +\infty$.

**Thm:** If $\sum a_k$ converges conditionally, the limit can be any $A \in \mathbb{R}$ by a reordering.

**Prop (Leibniz criterion):** Given $a_k \geq 0$, decreasing to $0$. Then $\sum (-1)^n a_n$ converges and
$$\sum_{k=0}^{2n+1} (-1)^k a_k \leq \sum_{k=0}^{\infty} (-1)^k a_k \leq \sum_{k=0}^{2n} (-1)^k a_k \quad \forall n \in \mathbb{N}.$$

**Prop (Cauchy Criterion):** $\sum a_k$ converges $\Leftrightarrow \forall \varepsilon > 0 \ \exists N: \left|\sum_{k=m+1}^{n} a_k\right| < \varepsilon \ \forall n > m \geq N$.

**Prop:** If $\sum a_k$ converges absolutely, it converges and $\left|\sum_{k=0}^{\infty} a_k\right| \leq \sum_{k=0}^{\infty} |a_k|$.

**Prop (Cauchy root criterion):** Consider $\sum a_k$ and let $\alpha := \limsup_{n \to \infty} \sqrt[n]{|a_n|}$. Then: $\alpha < 1 \Rightarrow$ absolute convergence; $\alpha > 1 \Rightarrow$ No Convergence.

**Prop (D'Alembert's quotient criterion):** Let $\alpha := \lim_{n \to \infty} \left|\frac{a_{n+1}}{a_n}\right|$. Then: $\alpha < 1 \Rightarrow$ absolute convergence; $\alpha > 1 \Rightarrow$ No Convergence.

**Prop:** If $\sum a_k$ converges absolutely, any reordering converges to the same value.

**Cor:** Let $\sum a_k, \sum b_k$ be absolutely convergent. Then $\sum a_k \cdot \sum b_k = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_{n-k} b_k\right)$.

**Def (Power Series):** $\sum_{k=0}^{\infty} a_k x^k$, $a_k, x \in \mathbb{R}$. (Generalization of Polynomials)

**Def (Radius of Convergence):** $R = \frac{1}{\limsup \sqrt[n]{|a_n|}}$.

**Thm:** Given $R \in (0, \infty)$, the series converges if $x \in (-R, R)$ and diverges if $|x| > R$

**Thm:** $\forall r < R$, $f_n(x) = \sum_{k=0}^{n} a_k x^k$ converges uniformly to $\sum_{k=0}^{\infty} a_k x^k$ on $[-r, r]$.

**Cor:** $\sum_{k=0}^{\infty} a_k x^k$ is continuous on $(-R, R)$.

**Def (Exponential):** $\exp(z) = \sum_{k=0}^{\infty} \frac{z^k}{k!} \ \forall z \in \mathbb{C}$.

**Thm:** $\exp$ is continuous and $e^{z+w} = e^z \cdot e^w$, $|e^z| = e^{Re(z)}$.

**Def (cos & sin):** $\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$, $\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

**Thm:** $\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$; $\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$; $\sin^2 + \cos^2 = 1$.

**Def (cosh & sinh):** $\cosh(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$, $\sinh(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$

**Def (Derivative)** $f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$.

**Lemma:** $f$ differentiable $\Rightarrow f$ continuous.

**Prop (Sum Rule):** $(f+g)' = f' + g'$.

**Prop (Product Rule):** $(f \cdot g)' = f'g + g'f$.

**Thm (Chain Rule):** $(g \circ f)' = (g' \circ f) \cdot f'$.

**Cor (Quotient Rule):** $\left(\frac{f}{g}\right)' = \frac{f'g - g'f}{g^2}$.

**Thm (Inverse Function):** $(f^{-1})' = \frac{1}{f' \circ f^{-1}}$.

**Prop:** $x_0$ left and right acc. pt, $f$ differentiable; $x_0$ extremum $\Rightarrow f'(x_0) = 0$

**Cor:** $x_0$ local extremum $\Rightarrow x_0$ endpoint of $I$, $f$ NOT differentiable, $f'(x_0) = 0$.

**Thm (Rolle):** $f: [a,b] \to \mathbb{R}$ cont, diff on $(a,b)$. If $f(a) = f(b)$, $\exists \xi \in (a,b): f'(\xi) = 0$.

**Thm (MVT):** $f: [a,b] \to \mathbb{R}$ cont, diff on $(a,b) \Rightarrow \exists \xi \in (a,b): f'(\xi) = \frac{f(b) - f(a)}{b - a}$

**Prop:** $f$ differentiable. $f$ Lipschitz $\Leftrightarrow f'$ bounded.

**Thm (L'Hôpital):** If $\lim_{x \to a} \frac{f}{g} = \frac{0}{0}$ or $\frac{\infty}{\infty}$, $\lim_{x \to a} \frac{f}{g} = \lim_{x \to a} \frac{f'}{g'}$.

**Prop:** $f: I \to \mathbb{R}$ differentiable. $f$ increasing $\Leftrightarrow f' \geq 0$.

**Def (Convexity):** $f: I \to \mathbb{R}$. $\forall a, b \in I \ \forall t \in (0,1): f(ta + (1-t)b) \leq tf(a) + (1-t)f(b)$

**Thm:** $f$ twice differentiable: $f$ convex $\Leftrightarrow f'$ increasing $\Leftrightarrow f'' \geq 0$.

Def (Partition): Collection $\mathcal{P}$ of subsets of $X$: $\bigcup_{A\in\mathcal{P}} A = X$ and $A \cap B = \emptyset$.

Def (Decomposition): Given $[a,b]$, $\{x_0, \dots, x_n\}$ s.t. $a = x_0 < \dots < x_n = b$.

Def (Step function): $f:[a,b] \to \mathbb{R}$ s.t. $\exists$ decomposition s.t. $f|_{(x_k, x_{k+1})} = \text{const}$.

Prop: $f,g:[a,b]\to\mathbb{R}$ step functions $\Rightarrow \alpha f + \beta g$ step function.

Def: $f:[a,b]\to\mathbb{R}$ step function. $\int_a^b f(x)dx = \sum_{k=1}^{n} c_k (x_k - x_{k-1})$, $c_k = f|_{(x_k - x_{k+1})}$.

Prop: $f,g:[a,b]\to\mathbb{R}$ step functions $\Rightarrow \int_a^b \alpha f + \beta g\, dx = \alpha \int_a^b f dx + \int_a^b g dx$.

Prop: $f,g:[a,b]\to\mathbb{R} \in \mathcal{SF}$, $f \leq g \Rightarrow \int_a^b f dx \leq \int_a^b g dx$.

Def (Riemann Integral): $f:[a,b]\to\mathbb{R}$ bounded. $\sup \mathcal{L}(f) = \inf \mathcal{U}(f) = \int_a^b f dx$.

Prop: $f,g:[a,b]\to\mathbb{R}$, R-integrable. $\int_a^b \alpha f + \beta g\, dx = \alpha \int_a^b f dx + \beta \int_a^b g dx$.

Thm: $f,g$ integrable. $f \leq g \Rightarrow \int_a^b f dx \leq \int_a^b g dx$.

Thm: $f:[a,b]\to\mathbb{R}$ integrable. $\left|\int_a^b f dx\right| \leq \int_a^b |f| dx$

Thm: Monotone Functions are integrable.

Thm: Continuous Functions are integrable.

Thm: $f_n:[a,b]\to\mathbb{R}$ integrable, uniformly converging to $f$: $\int_a^b f_n dx \to \int_a^b f dx$

Def (Primitive): $F$ is a primitive of $f$ if $F'(x) = f(x)$.

Thm: $f:[a,b]\to\mathbb{R}$ cont. $F(x) = \int_a^x f(t)dt + C$ for $C \in \mathbb{R}$ are all primitives.

Cor: $F:[a,b]\to\mathbb{R}$ differentiable $\Rightarrow F(x) = F(a) + \int_a^x F'(t)dt$.

Cor: $f:[a,b]\to\mathbb{R}$ cont, $F$ primitive of $f$. $\int_a^b f(x)dx = F(b) - F(a)$.

Thm: $\int_a^b f' g\, dx = [fg]_a^b - \int_a^b f g' dx$.

Thm: $\int_a^b g(f(x)) f'(x) dx = \int_{f(a)}^{f(b)} g(y) dy$.

Thm: $\int_a^b g(f(x)) dx = \int_{f(a)}^{f(b)} g(y) \cdot (f^{-1})'(y) dy$.

Thm: $f:[0,\infty) \to \mathbb{R}_{\geq 0}$ decreasing. $\sum_{k=1}^{\infty} f(k) \leq \int_0^\infty f(x)dx \leq \sum_{k=0}^{\infty} f(k)$

Thm: $f(x) = \sum_{n=0}^{\infty} a_n x^n$ with $R>0$. Then $F(x) = \sum_{n=0}^{\infty} a_n \int_0^x t^n dt = \sum_{n=0}^{\infty} a_n \frac{x^{n+1}}{n+1}$ has $R=R$.

Cor: $f(x) = \sum_{n=0}^{\infty} a_n x^n$ with $R>0$. Then $f'(x) = \sum_{n=0}^{\infty} n \cdot a_n x^{n-1}$ has $R=R$.

Cor: Power Series are smooth within their radius of convergence.

Def (Taylor Approximation): $f:I\to\mathbb{R}$ $n$ times differentiable. $P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$.

Thm (Integral Remainder): $n \geq 1$, $f:[a,b]\to\mathbb{R} \in C^n$. $f(x) = P_{n-1}(x) = \int_{x_0}^x f^{(n)}(t) \frac{(x-t)^{n-1}}{(n-1)!} dt$.

Thm (Lagrange Remainder): $n \geq 1$, $f:[a,b]\to\mathbb{R}$, $n$ times diff. $\exists \xi_L$: $f(x) = P_{n-1}(x) + \frac{f^{(n)}(\xi_L)}{n!}(x-x_0)^n$.

Thm: If $f \in C^n$, $f(x) - P_n(x) = o(|x-x_0|^n)$.

Thm: If $f$ $n$-times diff. and $|f^{(n)}| \leq M$, $f(x) - P_{n-1}(x) = O(|x-x_0|^n)$

Def (Analyticity): $f \to \mathbb{R}$ smooth, Power Series at $x_0$ has $R>0$, $\exists \delta > 0$: $f(x) = \sum_{k=0}^{\infty} f^{(k)}(x_0) \frac{(x-x_0)^k}{k!}$.

Thm: $f:I\to\mathbb{R} \in C^\infty$, $\exists\, r, C_0, A > 0$: $|f^{(n)}(x)| \leq C_0 A^n n!$ for $|x-x_0| < r$. $\Rightarrow f$ analytic at $x_0$.

Def (ODE): Relation between $x$, $u(x)$ and it's derivatives

Thm: $f:I-\mathbb{R}$. $u' + fu = 0$ has solution $u(x) = A \cdot e^{-F(x)}$.

Thm: $f,g:I\to\mathbb{R}$. $u' + fu = g$ has solution $u(x) = H(x)e^{-F(x)} + Ae^{-F(x)}$ with $H(x)$ primitive of $g(x)e^{F(x)}$.

Thm: $u'(x) = f(u(x))g(x)$ has solutions $u(x) = H^{-1}(G(x) + A)$ wit $H$ primitive of $\frac{1}{f}$.

Thm: $u'' + a_1 u' + a_0 u = 0$, $\Delta = a_1^2 - 4a_0$ has solutions: $\Delta = 0$: $e^{-\frac{a_1}{2}}(A + Bx)$, $\Delta > 0$: $u(x) = Ae^{\frac{-a_1 + \sqrt{\Delta}}{2}x} + Be^{\frac{-a_1 - \sqrt{\Delta}}{2}x}$, $\Delta < 0$: $e^{-\frac{a_1}{2}x}\left(A\cos\left(\frac{\sqrt{\Delta}}{2}x\right) + B\sin\left(\frac{\sqrt{\Delta}}{2}x\right)\right)$