Analysis II

Metric Spaces

Def (Euclidean Structure): $\|x\| = \sqrt{x_1^2 + \dots + x_n^2}$, $x \cdot y = \langle x, y \rangle = \sum_{i=1}^{n} x_i y_i$, $d(x,y) = \|x-y\|$

Lemma (Triangle Inequality): $\|z-x\| \leq \|z-y\| + \|y-x\|$ or $\|a+b\| \leq \|a\| + \|b\|$.

Lemma (Cauchy Schwarz): $x \cdot y \leq \|x\| \cdot \|y\|$.

Def (Metric Space): (X,d), $d: X \times X \to [0,\infty)$ with $d(x,y)=0 \Leftrightarrow x=y$, $d(x,y)=d(y,x)$,
$d(x,z) \leq d(x,y) + d(y,z)$.

Def (Sequence): $x: \mathbb{N} \to X$ written $(x_n)_{n \geq 0}$.

Def (Convergence): $\exists x \in X, \forall \varepsilon > 0\ \exists N \in \mathbb{N}: d(x_n, x) < \varepsilon\ \forall n \geq N$.

Lemma: The limit, if it exists, is unique.

Def (Accumulation Point): $Y \subset X$, $y \in X$ if $\exists (y_n) \subseteq Y$ s.t. $y_n \to y$ OR $(x_n) \subseteq X$ s.t.
$\exists (x_{n_k})_{k \geq 0}$ with $x_{n_k} \to x$

Lemma: (x_n) converges to $x \Leftrightarrow \forall (x_{n_k})$, $x_{n_k} \to x$.

Def (Cauchy Sequence): $\forall \varepsilon > 0\ \exists N$ s.t. $d(x_n, x_m) < \varepsilon\ \forall n,m \geq N$.

Def (Complete metric space): Every Cauchy Sequence converges.

Lemma: $(x_m)_{m \geq 0} \subset \mathbb{R}$. $x_m \to x \Leftrightarrow x_{m,i} \to x_i\ \forall i = 1, \dots, n$

Def (Open Ball): Given $r > 0$, $B(x,r) := \{y \in X \mid d(x,y) < r\}$.

Def (Open & Closed sets): $U \subset X$ open if $\forall x \in U\ \exists r > 0$, s.t. $B(x,r) \subseteq U$; closed if $X \setminus U$ open

Def (Interior): $\operatorname{int} \Omega = \Omega^\circ = \bigcup \{U \subset \Omega \mid U \text{ is open}\}$

Def (Closure): $\overline{\Omega} = \bigcap \{A \supset E \mid A \text{ is closed}\} = \{x \in X \mid \exists (x_n) \subseteq \Omega, x_n \to x\}$.

Def (Boundary): $\partial \Omega = \overline{\Omega} \setminus \Omega^\circ$.

Lemma: $U \subset X$ open $\Leftrightarrow x_n \to x \in U$, $x_n \in U$ eventually; closed $\Leftrightarrow (x_n) \subset A$ $x_n \to x \in A$.

Def (Continuity): $\forall \varepsilon > 0\ \exists \delta > 0: f(B(x,\delta)) \subseteq B(f(x), \varepsilon)$; $x_n \to x \Rightarrow f(x_n) \to f(x)$;
$\forall U \subseteq Y$ open, $f^{-1}(U)$ is open.

Def (Uniform continuity): $\forall \varepsilon > 0\ \exists \delta > 0\ \forall x,y$ $d(x,y) < \delta \Rightarrow d(f(x), f(y)) < \varepsilon$.

Def (Lipschitz continuity): $\exists L > 0\ \forall x,y$ $d(f(x), f(y)) \leq L \cdot d(x,y)$

Thm (Banach Fixed Point): $T: X \times X$ λ-Lipschitz with $\lambda \in (0,1) \Rightarrow \exists!\, x: T(x) = x$.

Def (Cover): $E \subset X$, $\mathcal{U} = \{U_i\}$ is a cover if $E \subset \bigcup \mathcal{U}$, $V \subset \mathcal{U}$ subcover

Def (Compact): $\forall (x_n)_{n \geq 0} \subset K\ \exists x_{n_k} \to x \in K$; $\forall$ open cover $\exists$ finite subcover

Cor: Compact $\Rightarrow$ closed, complete, closed subsets are compact

Prop: $f: X \to Y$ continuous, $K \subset X$ compact $\Rightarrow f(K)$ compact.

Thm: $f: X \to \mathbb{R}$ continuous, bounded, $K \subset X$ compact $\Rightarrow$ max/min are attained

Thm (Heine-Borel): $K \subset \mathbb{R}^n$ compact $\Leftrightarrow K$ closed & bounded.

Def (Connectedness): $A \subset X$ disconnected if $\exists U_1, U_2$ open, disjoint, $A \subset U_1 \cup U_2$,
$A \cap U_i$ nonempty. A is connected if not disconnected.

Prop: $E \subset \mathbb{R}$ connected $\Leftrightarrow E$ is an interval.

Prop: $f: X \to Y$ continuous, $E \subset X$ connected $\Rightarrow f(E)$ connected.

Cor (IVT): X connected, $f: X \to \mathbb{R}$ continuous, $f(x) \leq f(y) \Rightarrow \exists t \in X$ s.t. $f(t) = c\ \forall c \in [f(a), f(b)]$

Def (Curve): $\gamma: [0,1] \to X$ continuous. Closed if $\gamma(0) = \gamma(1)$.

Def (Path Connectedness): $E \subset X$, $\forall x,y \in E$, $\exists$ path, s.t. $\gamma(0) = x \wedge \gamma(1) = y$.

Prop: Path Connected $\Rightarrow$ Connected.

Thm: $U \subset \mathbb{R}^n$ open: Connected $\Leftrightarrow$ Path Connected.

Prop: $K \subset X$ compact, $f: K \to Y$ continuous $\Rightarrow$ uniformly continuous.

Def (Norm): $\|\cdot\|: V \to [0,\infty)$ s.t. $\|v\| = 0 \Leftrightarrow v = 0$, $\|\alpha v\| = |\alpha| \|v\|$, $\|v+u\| \leq \|v\| + \|u\|$

Def (Hilbert-Schmidt norm): $\|M\|_2 = \sqrt{tr(M^T \cdot M)}$

Lemma: $\forall x \in \mathbb{R}^n$, $|M \cdot x| \leq \|M\|_2 \cdot |x|$

Prop: Norm $\Rightarrow$ Metric with $d(x,y) = \|x-y\|$

Def (Inner Product): $\langle \cdot, \cdot \rangle: V \times V \to \mathbb{R}$ s.t. $\langle u,v \rangle = \langle v,u \rangle$, $\langle \alpha u + \beta v, w \rangle = \alpha \langle u,w \rangle + \beta \langle v,w \rangle$
and $\langle v,v \rangle \geq 0$, $\langle v,v \rangle = 0 \Leftrightarrow v = 0$.

Prop: Inner Product $\Rightarrow$ Norm with $\|v\| = \sqrt{\langle v,v \rangle}$.

Multidimensional Differentation

Def (Differential): $Df_{x_0} = Df(x_0)$ linear, such that $\lim_{x \to 0} \frac{f(x_0+x) - f(x_0) - L(x)}{|x|} = 0$.

Lemma: $f: U \to \mathbb{R}^m$ differentiable $\Leftrightarrow f_i: U \to \mathbb{R}$ differentiable $\forall i$. Then $Df_{x_0}(x) = \begin{pmatrix} Df_{1,x_0}(x) \\ \vdots \\ Df_{m,x_0}(x) \end{pmatrix}$.

Def (Directional Derivative): $\partial_v f(x_0) = \lim_{t \to 0} \frac{f(x_0 + tv) - f(x_0)}{t}$.

Prop: f differentiable at $x_0 \Rightarrow \partial_v f(x_0) = Df_{x_0}(v)$ exists.

Def (Partial derivative): $\partial_i f(x_0) = \frac{\partial f}{\partial e_i}(x_0) := \partial_{e_i} f(x_0)\ \forall i \in \{1, \dots, n\}$.

Thm: $\partial_i f(x)$ exists and continuous $\forall i \Rightarrow f$ differentiable, $Df_{x_0}(x) = \begin{pmatrix} \partial_1 f(x_0) & \cdots & \partial_n f(x_0) \end{pmatrix} \cdot x$.

Def (C^k): $C^0(U, \mathbb{R}^m) := \{f: U \to \mathbb{R}^m \mid f \text{ cont.}\}$; $C^k := \{f \in C^{k-1} \mid \partial_i f \in C^{k-1}\}$

Def (Jacobi matrix): $Jf(x_0) := Df(x_0) = \begin{pmatrix} \partial_1 f_1(x_0) & \cdots & \partial_n f_1(x_0) \\ \vdots & \ddots & \vdots \\ \partial_1 f_m(x_0) & \cdots & \partial_n f_m(x_0) \end{pmatrix}$

Thm (Chain rule): $D(g \circ f)_{x_0} = Dg_{f(x_0)} \circ Df_{x_0}$; $\frac{\partial (g \circ f)_k}{\partial x_i}(x) = \sum_{j=1}^{m} \frac{\partial g_k}{\partial y_j}(f(x)) \cdot \frac{\partial f_j}{\partial x_i}(x)$

Thm (Mean Value Theorem): $f \in C^1(U)$, $x_0 + t \cdot h \in U\ \forall t \in [0,1] \Rightarrow \exists \theta \in [0,1]$ s.t. $f(x_0+h) - f(x_0) = Df_{x_0 + \theta h}(h)$

Def (Convex Set): $A \subset \mathbb{R}^n$ s.t. $\forall x,y \in A$, $t \in [0,1]$, $tx + (1-t)y \in A$.

Def (Gradient): $f \in C^1(U)$, $\nabla f(x) = Jf(x)^T = \begin{pmatrix} \partial_1 f(x) \\ \vdots \\ \partial_n f(x) \end{pmatrix}$

Thm: $U \subseteq \mathbb{R}^n$, $f \in C^1(U, \mathbb{R}^m)$, $\Lambda := \sup_{x \in U} \|Jf(x)\|_2 < \infty \Rightarrow f$ is $\sqrt{m}\lambda$-Lipschitz.

Thm: $f: U \to \mathbb{R}^m$ diff., $Df(x) = U\ \forall x \in U \Rightarrow f$ constant on connected components.

Thm (Schwarz): $f \in C^2(U, \mathbb{R}^m) \Rightarrow \partial_i \partial_j f = \partial_j \partial_i f$

Def (Multi-Index): $\alpha := (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, $|\alpha| = \alpha_1 + \dots + \alpha_n$, $\alpha! = \alpha_1! \cdots \alpha_n!$.

Def: $\partial^\alpha f = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} f$, $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$

Thm (Taylor): $f \in C^{k+1}(U)$, $x_0 \in U$, $h \in \mathbb{R}^n$ s.t. $x_0 + th \in U\ \forall t \in [0,1] \Rightarrow f(x_0+h) = \sum_{|\alpha| \leq k} \partial^\alpha f(x_0) \frac{h^\alpha}{\alpha!} + O(h^{k+1})$

Def (Real analytic): $f \in C^\infty(U)$, $\forall x_0 \in U\ \exists \rho > 0$, $C > 0$ s.t. $\max_{B_\rho(x_0)} |\partial^\alpha f| \leq C |\alpha|! (n\rho)^{-|\alpha|}\ \forall \alpha \in \mathbb{N}^n$.

Thm: f analytic $\Rightarrow \exists \rho > 0$ s.t. $f(x_0 + h) = \sum_{|\alpha|=1}^{\infty} \partial^\alpha f(x_0) \frac{h^\alpha}{\alpha!}\ \forall h \in B_\rho(0)$

Cor: U connected, f, g analytic s.t. $\exists x_0$ with $\partial^\alpha f(x_0) = \partial^\alpha g(x_0)\ \forall \alpha \Rightarrow f = g$.

Optimization

Def (Local Minimum): $\exists r > 0: f(x_0) \leq f(x)\ \forall x \in B_r(x_0)$.

Def (Critical Point): $f \in C^1$, $\nabla f(x_0) = 0$.

Prop: Local Extremum $\Rightarrow$ Critical Point

Thm (Lagrange Multipliers): $M = \{x \in U \mid g_1(x) = \dots = g_m(x) = 0\}$. For x_0 local extremum of $f|_M$
then $\{\nabla f(x_0), \nabla g_1(x_0), \dots, \nabla g_m(x_0)\}$ are linearly dependent.

Thm (Spectral Theorem): $A \in M_{n \times n}(\mathbb{R})$ symmetric diagonalizes in an ONB

Def (Hessian): $H_{ij} f(x) = \partial_i \partial_j f(x)$. $Hf(x) = D^2 f(x)$. $\Delta f(x) = tr(Hf(x)) = \sum_{i=1}^{n} \partial_{ii} f(x)$.

Prop: x_0 loc. min. $\Rightarrow Hf(x_0)$ non-neg def n; $Hf(x_0)$ pos. def n. $\Rightarrow x_0$ loc. min.

Def (Convex): $A \subseteq \mathbb{R}^n$ convex, $f: A \to \mathbb{R}$ s.t. $\forall x,y \in A$, $t \in [0,1]$, $f((1-t)x + ty) \leq (1-t)f(x) + tf(y)$.

Inverse & Implicit Function Theorem

Lemma: ϕ contraction, $F = x + \phi(x)$, $\forall x_0 \in U, r > 0$ s.t. $B_r(x_0) \subseteq U$, $B_{(1-\lambda)r}(F(x_0)) \subseteq F(B_r(x_0)) \subseteq F(U)$. $F(U)$ open, $F: U \to F(U)$ bijective. F^{-1} is $\frac{1}{1-\lambda}$-Lipschitz.

Lemma: f differentiable, Df_{x_0} invertible, g Lipschitz, $f \circ g(y) = y$. $\Rightarrow g$ differentiable at $y_0 = f(x_0)$, $Dg_{y_0} = (Df_{x_0})^{-1}$.

Thm (Inverse Fct.): $f \in C^1(U)$, Df_{x_0} invertible for $x_0 \in U \Rightarrow \exists U_0 \subseteq U$ s.t. $f|_{U_0}$ injective, $f(U) =: V$ open, $g = f|_{U_0}^{-1}: V \to U_0 \in C^1$, $Dg_{f(x)} = (Df_x)^{-1}$ $\forall x \in U_0$.

Def (Diffeomorphism): $f: U \to V \in C^1$ s.t. $f^{-1}: V \to U \in C^1$.

Thm (Implicit Fct.): $0 < d < n$, $k \geq 1$. $f \in C^1(U, \mathbb{R}^{n-d})$ s.t. $\exists (\underset{\mathbb{R}^d}{x_0}, \underset{\mathbb{R}^{n-d}}{y_0}) \in U$ with $f(x_0, y_0) = 0$, $J_y f_{(x_0, y_0)}$ invertible $\Rightarrow \exists r, s > 0$, $g: B_r(x_0) \to B_s(y_0)$ s.t. $f(x,y) = 0 \Leftrightarrow y = g(x)$ $\forall (x,y) \in B_r(x_0) \times B_s(y_0)$, $Jg(x) = -((J_y f)(x, g(x)))^{-1} \cdot (J_x f(x, g(x)))$.

Submanifolds

Def (Submanifold): $0 < d < n$, $k \geq 1$, $M \subset \mathbb{R}^n$ s.t. $\forall p_0 \in M$ $\exists U \subseteq \mathbb{R}^n$ containing p_0, $V \subseteq \mathbb{R}^n$ containing 0, $\Psi: U \to V$ C^k-diffeomorphism s.t. $\Psi(M \cap U) = \{y \in V \mid y_{d+1} = \cdots = y_n = 0\}$

Def (Implicit): $\forall p \in M$ $\exists U \subseteq \mathbb{R}^n$ containing p, $f \in C^k(U, \mathbb{R}^{n-d})$ s.t. $rk(Df_p) = n-d$, $M \cap U = f^{-1}(0)$.

Def (Parametric): $\forall p \in M$ $\exists V \subseteq \mathbb{R}^d$, $q \in V$, $G: C^k(V, \mathbb{R}^n)$ s.t. $rk(DG_q) = d$, $G(q) = p$ and $\forall V' \subset V$ $\exists U' \subset \mathbb{R}^n$ with $M \cap U' = G(V')$

Def (Graphical): $\forall p \in M$ $\exists U \subseteq \mathbb{R}^n$ containing p, $V \subset \mathbb{R}^d$, $g \in C^k(V, \mathbb{R}^{n-d})$ s.t. $M = \{x \in \mathbb{R}^n \mid (x_{d+1}, \dots, x_n) = g(x_1, \dots, x_d)\}$.

Thm: The 4 definitions are equivalent.

Def (Tangent vector): $\tau \in \mathbb{R}^n$ s.t. $\exists p_k \in M, r_k > 0$ with $\frac{p_k - p}{r_k} \to \tau$ when $p_k \to p$, $r_k \to 0$.

Def (Normal vector): $\nu \in \mathbb{R}^n$ s.t. $\nu \cdot \tau = 0$ $\forall \tau \in T_pM$.

Lemma: $T_pM = D\phi_q(\mathbb{R}^d)$ $\forall \phi: V \to M \cap U$ parametrization

Multidimensional Integration

Def (Dyadic Cube): $Q \subseteq \mathbb{R}^n$ s.t. $Q = 2^{-\rho}(\alpha + [0,1)^n)$ for $\alpha \in \mathbb{Z}^n$.

Def (Volume): $F = \bigcup_{i=1}^N 2^{-\rho}(\alpha_i + [0,1)^n)$. $\mu(F) := N \cdot 2^{-\rho \cdot n}$

Def (Inner/Outer Volume): $\mu_{out}(E) = \inf\{\mu(G) \mid E \subseteq G, G \text{ dyadic}\}$, $\mu_{in}(E) = \sup\{\mu(F) \mid F \subseteq E, F \text{ dyadic}\}$

Def (Jordan Measurable): $E \subseteq \mathbb{R}^n$ bounded, s.t. $\mu_{out}(E) = \mu_{in}(E) =: \mu(E) = vol_n(E)$.

Lemma: $E = (a_1, b_1) \times \cdots \times (a_n, b_n)$, $\bar{E}$ are Jordan measurable, $\mu(E) = \mu(\bar{E}) = \prod_{i=1}^n (b_i - a_i)$.

Prop: E, F J.M. $\Rightarrow E \cup F$, $E \cap F$, $E \setminus F$ J.M.

Lemma: $\forall \varepsilon > 0$ $\exists F, G$ J.M. with $F \subset E \subset G$, $\mu(G) - \mu(F) < \varepsilon \Rightarrow E$ J.M.

Def (Jordan Null): $E \subseteq \mathbb{R}^n$ with $\mu_{out}(E) = 0$.

Thm: $E \subseteq \mathbb{R}^n$ J.M. $\Leftrightarrow$ E bdd & ∂E null.

Lemma: E null, $f: E \to \mathbb{R}^n$ Lipschitz $\Rightarrow f(E)$ null.

Prop: E J.M. $\Rightarrow \rho E + a$ J.M., $\mu(\rho E + a) = \rho^n \mu(E)$.

Thm: $L: \mathbb{R}^n \to \mathbb{R}^n$ linear. E J.M. $\Rightarrow L(E)$ J.M, $\mu(L(E)) = |\det(L)| \cdot \mu(E)$.

Def (Dyadic Step Fct.): $g(x) = \sum_{\ell=1}^N g_\ell \mathbb{1}_{Q_\rho(a_\ell)}(x)$

Def: $\int g(x) = \sum_{\ell=1}^N g_\ell \cdot \mu_n(Q_\rho(a_\ell)) = 2^{-\rho n} \sum_{\ell=1}^N g_\ell$.

Def (Riemann Integrable): $\mathcal{L}_A := \sup\{\int g \mid g \leq f \cdot \mathbb{1}_A, g \text{ S.F.}\} = \inf\{\int g \mid g \geq f \cdot \mathbb{1}_A, g \text{ S.F.}\} =: \mathcal{U}_A$

Lemma: f, g R.I. $\Rightarrow c_1 f + c_2 g$ R.I., $\int_A (c_1 f + c_2 g) = c_1 \int_A f + c_2 \int_A g$.

Lemma: f R.I $\Leftrightarrow f^+, f^-$ R.I. In particular $\int_A f = \int_A f^+ + \int_A f^-$.

Lemma: $\mu_{n+1}(\Gamma_f) = \int_A f$ (Γ_f Hypograph)

Lemma: $f: A \to [0, \infty)$ uniformly cont. $\Rightarrow f$ R.I.

Thm (Change of Variables): $\int_A f(x)\,dx = \int_{\Phi(A)} \frac{f(\Phi^{-1}(y))}{|\det J\Phi(\Phi^{-1}(y))|}\,dy$.

Thm: $\int_A f(x)\,dx = \int_{\Psi^{-1}(A)} f(\Psi(y)) |\det J\Psi(y)|\,dy$.

Thm (Cavalieri): A J.M., $A \subseteq (-C, C)^n$. $A_x := \{y \in \mathbb{R}^{n-k} \mid (x,y) \in A\}$. $\bar{\varphi}(x) = \mu_{n-k,out}(A_x)$ $\underline{\varphi}(x) = \mu_{n-k,in}(A_x) \Rightarrow \bar{\varphi}, \underline{\varphi}$ R.I., $\int_{(-C,C)^k} \underline{\varphi}(x) = \mu_n(A) = \int_{(-C,C)^k} \bar{\varphi}(x)$.

Thm (Fubini): A bdd. $\tilde{f} = f \cdot \mathbb{1}_A$. $\underline{\varphi}(x) = \underline{\int} \tilde{f}_x(y)\,dy$, $\bar{\varphi}(x) = \overline{\int} \tilde{f}_x(y)\,dy \Rightarrow \int_A f = \int_{\mathbb{R}^k} \underline{\varphi}(x) = \int_{\mathbb{R}^k} \bar{\varphi}(x)$

Cor: $K = [a_1, b_1] \times \cdots \times [a_n, b_n]$, $f: K \to \mathbb{R}$ cont. $\Rightarrow f$ R.I., $\int_K f = \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} f(x_1, \dots, x_n)\,dx_n \cdots dx_1$

Thm (Differentiation under $\int$): K cpt, $f \in C^0(K \times [a,b]) = f(x,t)$, $\partial_t f \in C^0 \Rightarrow \frac{d}{dt}\big|_{t_0} \int_K f = \int_K \partial_t f(x, t_0)\,dx$.

Def (Improper $\int$): U open, f cont. $\int_U f = \sup\{\int_K f \mid K \subset U, K \text{ cpt}, \text{J.M}\}$.

Def (d-volume): $\phi: V \to \mathbb{R}^n$ d dim submanifold. $E \subset V$ J.M, $\bar{E} \subseteq V \Rightarrow vol_d(\phi(E)) = \int_E \sqrt{\det((D\phi_x)^T D\phi_x)}\,dx$

Def (Support): $spt(f) = \overline{\{x \in V \mid f(x) \neq 0\}} \subseteq \bar{V} \subseteq \mathbb{R}^m$.

Def: $\phi: V \to \mathbb{R}^n$ d dim submanifold, $f: \phi(V) \to \mathbb{R}$ cont., $spt(f) \subseteq V \Rightarrow \int_{\phi(V)} f(p)\,dvol_d(p) = \int_V (f \circ \phi) \sqrt{\det((D\phi)^T D\phi)}\,dx$

Lemma: $\phi(x') = (x', g(x')) \Rightarrow \sqrt{\det((D\phi)^T D\phi)} = \sqrt{1 + |\nabla g(x')|^2}$.

Def: $\nu_p = \nu(x') := \frac{(-\nabla g(x'), 1)}{\sqrt{1 + |\nabla g(x')|^2}}$

Lemma: ν is normal to M, $|\nu_p| = 1$ and ν is pointing outwards.

Lemma (Graphical div. thm): $V \subseteq \mathbb{R}^{n-1}$ open, $g \in C^1(V, \mathbb{R})$, $\Omega = B_r(0) \cap \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \mid x_n < g(x')\}$, $M := \{(x', g(x'))\}$. $f \in C^1(\bar{\Omega})$, $F := f e_n$, $spt(f) \subseteq \bar{\Omega} \cap B_r(0) \Rightarrow \int_\Omega \partial_n f\,dx = \int_M f(p) \nu(p) \cdot e_n\,dvol_M(p)$.

Def (Divergence): $\operatorname{div} F := \sum_{i=1}^n \partial_i F_i$.

Thm (Local div thm): As above, $F \in C^1(\bar{\Omega}, \mathbb{R}^n)$ s.t. $spt(F) \subseteq \bar{\Omega} \cap B_r(0) \Rightarrow \int_\Omega \operatorname{div} F\,dx = \int_M F \cdot \nu\,dvol_M$

Def (C^k-domain): Bounded open set $\Omega \subset \mathbb{R}^n$ if $M := \partial\Omega$ is a $(n-1)$dim sbmfld of class C^k.

Thm (Divergence Thm): $\Omega \subseteq \mathbb{R}^n$ bdd C^1 domain, $G \in C^1(\bar{\Omega}, \mathbb{R}^n) \Rightarrow \int_\Omega \operatorname{div} G\,dx = \int_{\partial\Omega} G \cdot \nu\,dvol_{\partial\Omega}$.

Thm (Green): $\int_{\partial\Omega} F \cdot \tau\,dL = \int_\Omega \operatorname{rot} F\,dx$ ($\operatorname{rot} F = \partial_1 F_2 - \partial_2 F_1$).

Differential Forms

Def (1-Form): $\alpha: U \to (\mathbb{R}^n)^*$, $x \mapsto (\alpha_1(x), \dots, \alpha_n(x)) = \alpha_x$.

Ex (Associated 1-Form): $F \in C^1(U, \mathbb{R}^n) \Longrightarrow \alpha_F$ by $\alpha_x = F(x)^T$.

Def: $f \in C^\infty(U, \mathbb{R})$, $df: U \to (\mathbb{R}^n)^*$, $df_x = Df_x$.

Def (Line Integral): $\alpha: U \to (\mathbb{R}^n)^*$ C^1, $\gamma: [a,b] \to U$. $\int_\gamma \alpha := \int_a^b \alpha_{\gamma(t)}(\gamma'(t))\,dt = \int_a^b \alpha(\gamma(t)) \cdot \gamma'(t)\,dt$.

Def (Line Integral): $\alpha = F^T$. $\int_\gamma \alpha = \int_\gamma F^T = \int_a^b F(\gamma(t)) \cdot \gamma'(t)\,dt$.

Def (Conservative): U connected, $F: U \to \mathbb{R}^n$ if $\exists f \in C^2(U, \mathbb{R})$ s.t. $\nabla f = F$.

Def (Exact): U connected, $\alpha: U \to (\mathbb{R}^n)^*$ if $\exists f \in C^2(U, \mathbb{R})$ s.t. $df = \alpha$.

Lemma: α Exact $\Rightarrow \int_\gamma \alpha = f(\gamma(b)) - f(\gamma(a))$

Lemma (Integrability Criteria): α exact $\Leftrightarrow \partial_j \alpha_i = \partial_i \alpha_j$.

Def (k-form): $\alpha_p: \mathbb{R}^{n \times k} \to \mathbb{R}$ s.t. $\alpha_p(T \cdot S) = \alpha_p(T) \cdot \det(S)$ & α_p is k-linear.

Lemma: $\alpha(T) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} c_{i_1, \dots, i_k}(p) \cdot \det(E^T_{i_1, \dots, i_k} \cdot T) = \sum_{i \in \mathcal{I}_k} c_i(p) \cdot \det(E_i^T \cdot T)$. $\left(E_i^T = \begin{pmatrix} 0&0&1&0&0&0&0 \\ 0&0&0&0&1&0&0 \\ 0&0&0&0&0&1&0 \end{pmatrix}\right)$

Def (Wedge Product): $\alpha = \det(AT)$, $\beta = \det(BS) \Rightarrow \alpha \wedge \beta(X) := \det\left(\begin{bmatrix} A \\ B \end{bmatrix} \cdot X\right)$. (Extend Bilinear).

Def: 0-form $= C^\infty$ functions. k-forms $= \sum_{I \in \mathcal{I}_k} c_I(x) \cdot \underbrace{\det(E_I^T T)}_{:= w_I} = \alpha_x(T)$.

Def (Differential): $d\alpha_x(S) := \sum_I dc_I(x) \wedge w_I(S)$

Rem: $f(x) = x_i \Rightarrow dx_i = df_x = Df_x = e_i^T \Rightarrow w_i = dx_{i_1} \wedge \cdots \wedge dx_{i_k} =: dx_I$.

Lemma: $\forall \alpha \in \Omega^k(U)$, $d(d\alpha) = 0 \in \Omega^{k+2}(U)$

Lemma (Leibniz): $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta$

Cor: $\alpha = df_1 \wedge \cdots \wedge df_k \Rightarrow d\alpha = 0$.

Def (Pull-back): $\alpha \in \Omega^k(U)$, $V \subseteq \mathbb{R}^m$, $F \in C^\infty(V, U) \Rightarrow (F^*(\alpha))_x(A) = \alpha_{F(x)}(DF_x \cdot A) \in \Omega^k(V)$.

Lemma: $\alpha = e_I^*$, $F(y) = (F_1(y), \dots, F_n(y))^T \Rightarrow F^*(\alpha) = dF_{i_1} \wedge \dots \wedge dF_{i_k}$.

Prop: $F^*(\alpha \wedge \beta) = F^*(\alpha) \wedge F^*(\beta)$, $F^*(d\alpha) = d F^*(\alpha)$.

Def: $\Phi: V \to U$ param. submfld. $\alpha \in \Omega^k(U)$. $\int_M \alpha := \int_V \alpha_{\Phi(x)}(D\Phi_x)\,dx = \int_V (\Phi^*\alpha)_x\,dx$

Def (Orientable): Covered by local params with transition maps having $\det > 0$.

Prop: $(n-1)$ sbmfld orientable $\Leftrightarrow \exists\, \nu: M \to S^{n-1}$ s.t. $\nu(p) \perp T_pM$, ν continuous.

Thm: $\forall \alpha \in \Omega^{n-1}(\mathbb{R}^n)$, $\int_U d\alpha = \int_{\partial U} \alpha$. ($\partial U$ oriented according exterior unit normal)

Thm: M parametrized sbmfld with boundary, $\alpha \in \Omega^{m-1}(\mathbb{R}^n) \Rightarrow \int_{\partial M} \alpha = \int_M d\alpha$.

Cor: $\Sigma \subseteq \mathbb{R}^3$ 2d-surface, $F \in C^\infty(\mathbb{R}^3, \mathbb{R}^3) \Rightarrow \int_\Sigma \operatorname{rot} F \cdot \nu\, ds = \int_{\partial\Sigma} F \cdot dL$.

Def (Linear Const Coef ODE): $y^{(m)} + a_{m-1} y^{(m-1)} + \dots + a_1 y' + a_0 y = f \quad \forall t \in I$.

Def (Characteristic Polynomial): $p(x) = \sum_{k=0}^{m} a_k x^k \Rightarrow$ ODE: $Ly = f$ with $L = p\left(\frac{d}{dt}\right)$.

Def (Homogenous ODE): $Lz = 0$.

Lemma: α root of p with multiplicity $m \Rightarrow t^k \exp(\alpha_j \cdot t)$ $\forall k = 0, \dots, m$ solve the ODE

Prop: The solutions to $Lz = 0$ form a vector space with basis functions above.

Prop: $\begin{cases} Lz = 0 & \forall t \in I \\ z^{(k)}(t_0) = w_k & \forall k = 0, \dots, m-1 \end{cases}$ has a unique solution.

Prop: $Ly = f \Rightarrow$ Solutions are $\{y_{part} + z \mid z \in V\}$

Lemma: If $f(t) = q(t)e^{\omega t}$, $\deg(q) = m$: ω not root of $p \Rightarrow y(t) = Q(t)e^{\omega t}$ ($\deg Q = m$); ω root of p mult $k \Rightarrow y(t) = Q(t) \cdot t^k e^{\omega t}$ ($\deg Q = m$).

Lemma (Superposition): $Ly_i = f_i \Rightarrow L(y_1 + y_2) = f_1 + f_2$.

Def (Pulse at 0): $\begin{cases} Lz = 0 \\ z^{(k)} = 0 & \forall k = 0, \dots, m-2 \\ z^{(m-1)} = 1 \end{cases}$ Sol $P(t)$ (Pulse at 0).

Thm (Duhamel): $\begin{cases} Ly = f \\ y(t_0) = \dots = y^{(m-1)}(t_0) = 0 \end{cases} \Rightarrow y(t) = \int_{t_0}^{t} f(s) P(t-s)\,ds$.

Def (System of ODE's): $u'(t) = F(t, u(t))$.

Def (Constant Coef): $F(t, u(t)) = A \cdot u(t)$.

Def (Matrix Exponential): $\exp(A) := \sum_{k=0}^{\infty} \frac{A^k}{k!}$.

Thm: The unique sol of $\begin{cases} u' = Au \\ u(t_0) = x_0 \end{cases}$ is $u(t) = \exp(A(t - t_0)) \cdot x_0$.

Thm (Cauchy-Lipschitz): $F: U \to \mathbb{R}^n$, $F = F(t, x)$, C^0 in t, C^1 in $x \Rightarrow \exists$ maximal interval I,
$u \in C^1$ s.t. $\begin{cases} u'(t) = F(t, u(t)) & \forall t \in I \\ u(t_0) = x_0. \end{cases}$

Thm (Ascoli-Arzelà): $K \subseteq \mathbb{R}^n$ cpt, $f_k: K \to \mathbb{R}^m \in C_b(K, \mathbb{R}^m)$ s.t. $\sup_k \|f_k\|_\infty < \infty$,
$\forall \varepsilon > 0\ \exists \delta > 0$ s.t. $\forall k\ \forall x \in K$, $f(B_\delta(x) \cap K) \subseteq B_\varepsilon(f(x)) \Rightarrow \exists$ convergent subseq in $C_b(K, \mathbb{R}^n)$.