Mathematical Methods

Problem Set

The following problems are designed to help you practice the key methods covered in the lecture. To master the techniques presented it is nevertheless essential to do practice beyond this sheet, and to apply the methods to problems in other contexts. You for sure can make use of them in some other lectures this week.

Problem 1 Linearisation

Linearise the following functions around the specified points:

a) $f(x) = \sin(x)$ around $x_0 = \pi/4$

b) $f(x) = \ln(x)$ around $x_0 = 1$

c) $f(x) = e^x$ around $x_0 = 3$

d) $f(x) = \sqrt{x}$ around $x_0 = 4$

e) $f(x) = \sin(\ln(x))$ around $x_0 = 1$

f) $f(x) = \cos(x^3 - 2x) + \sqrt{x^2 + 1}$ around $x_0 = 5$

Solution to Problem 1

a) We have $f(\frac{\pi}{4}) = \sin(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$ and $f'(\frac{\pi}{4}) = \cos(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$. Therefore, the linearisation is:

$$L(x) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}\left(1 + x - \frac{\pi}{4}\right).$$

b) We have $f(1) = \ln(1) = 0$ and $f'(1) = \frac{1}{1} = 1$. Therefore, the linearisation is:

$$L(x) = 0 + 1(x - 1) = x - 1.$$

c) We have $f(3) = e^3$ and $f'(3) = e^3$. Therefore, the linearisation is:

$$L(x) = e^3 + e^3(x - 3) = e^3(1 + x - 3) = e^3(x - 2).$$

d) We have $f(4) = \sqrt{4} = 2$ and $f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$. Therefore, the linearisation is:

$$L(x) = 2 + \frac{1}{4}(x - 4) = 2 + \frac{1}{4}x - 1 = \frac{1}{4}x + 1.$$

e) We have $f(1) = \sin(\ln(1)) = \sin(0) = 0$ and $f'(1) = \cos(\ln(1)) \cdot \frac{1}{1} = 1$. Therefore, the linearisation is:

$$L(x) = 0 + 1(x - 1) = x - 1.$$

f) We have $f(5) = \cos(5^3 - 2 \cdot 5) + \sqrt{5^2 + 1} = \cos(125 - 10) + \sqrt{26} = \cos(115) + \sqrt{26}$ and $f'(5) = -\sin(115) \cdot (3 \cdot 5^2 - 2) + \frac{5}{\sqrt{26}}$. Therefore, the linearisation is:

$$L(x) = \cos(115) + \sqrt{26} + \left(-\sin(115) \cdot (3 \cdot 5^2 - 2) + \frac{5}{\sqrt{26}}\right)(x - 5).$$

Problem 2 **Numerical Differentiation**

Given is the function $f(x) = x^3 - x - 1$. Find the real root of this polynomial using both the bisection method and Newton's method.

Solution to Problem 2

The real solution of the polynomial $f(x) = x^3 - x - 1$ can be found using numerical methods.

$$x_0 \approx 1.3247.$$

Problem 3 **Integration**

Find the area under the curve of $f(x) = x^2$ from $x = 0$ to $x = 3$ using integration and the trapezoidal rule. Show that as the number of trapezoids increases, the approximation converges to the exact value.

Solution to Problem 3

The exact area under the curve of $f(x) = x^2$ from $x = 0$ to $x = 3$ is given by the definite integral:

$$A = \int_0^3 x^2 \, dx = \left[\frac{x^3}{3}\right]_0^3 = \frac{27}{3} - 0 = 9.$$

Using the trapezoidal rule with n trapezoids, the approximation is:

$$A_n = \frac{b-a}{2n} \left(f(a) + 2 \sum_{i=1}^{n-1} f(x_i) + f(b) \right).$$

As n increases, A_n converges to the exact value of 9. Some numerical results for different values of n are:

- $n = 1$: $A_1 = 13.5$
- $n = 2$: $A_2 = 10.125$
- $n = 4$: $A_4 = 9.28125$

Problem 4 **Statistics**

A dataset consists of the following values: $2, 4, 4, 4, 5, 5, 7, 9$. Calculate the mean, median, mode and standard deviation of the dataset. Furthermore, draw a boxplot to visualize the distribution of the data.

Solution to Problem 4

The mean is calculated as:

$$\text{Mean} = \frac{2+4+4+4+5+5+7+9}{8} = \frac{40}{8} = 5.$$

The median is the middle value of the ordered dataset:

$$\text{Median} = \frac{4+5}{2} = 4.5.$$

The mode is the value that appears most frequently:

$$\text{Mode} = 4.$$

The standard deviation is calculated as:

$$\begin{aligned}\sigma &= \sqrt{\frac{(2-5)^2+(4-5)^2+(4-5)^2+(4-5)^2+(5-5)^2+(5-5)^2+(7-5)^2+(9-5)^2}{8}} \\ &= \sqrt{\frac{9+1+1+1+0+0+4+16}{8}} = \sqrt{\frac{32}{8}} = 2.\end{aligned}$$

The boxplot can be drawn using the five-number summary: minimum = 2, Q1 = 4, median = 4.5, Q3 = 6, maximum = 9.

Problem 5 **Error Propagation**

Consider the function $f(x,y) = x^2 + y^2$. If $x = 3.0 \pm 0.1$ and $y = 4.0 \pm 0.2$, calculate the propagated uncertainty in f using the error propagation formula.

Solution to Problem 5

The propagated uncertainty in f can be calculated using the error propagation formula:

$$\sigma_f = \sqrt{\left(\frac{\partial f}{\partial x}\sigma_x\right)^2 + \left(\frac{\partial f}{\partial y}\sigma_y\right)^2}.$$

First, we calculate the partial derivatives:

$$\frac{\partial f}{\partial x} = 2x, \quad \frac{\partial f}{\partial y} = 2y.$$

Evaluating at $x = 3.0$ and $y = 4.0$:

$$\frac{\partial f}{\partial x} = 2(3.0) = 6.0, \quad \frac{\partial f}{\partial y} = 2(4.0) = 8.0.$$

Now, substituting the uncertainties:

$$\sigma_f = \sqrt{(6.0 \cdot 0.1)^2 + (8.0 \cdot 0.2)^2} = \sqrt{(0.6)^2 + (1.6)^2} = \sqrt{0.36 + 2.56} = \sqrt{2.92} \approx 1.71.$$